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An Introduction to Group Theory and to Unitary Symmetry Models

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Contents

1. General Notions on Groups [1]

1.1. A group G is a set of elements together with a multiplication law which associates a third element to any pair of elements of G: $(a, b) \to (ab) = c$ in such a way that the following conditions hold:

i) associativity $(ab)c = a(bc) = abc$; $a, b, c \in G$; [1])
ii) there exists an unit element $e \in G$ such that for any $a \in G$: $ae = ea = a$;
iii) for any $a \in G$ there exists an element a^{-1}, called the inverse of a, such that:

$$aa^{-1} = a^{-1}a = e.$$

If $ab = ba$ for every pair of elements in G, the group is said to be commutative or Abelian.

1.2. A subset G' is called a subgroup if the set of its elements is by itself a group under the multiplication law of G. It is easy to see that a subset G' of G is a subgroup if and only if $ab^{-1} \in G'$ for any pair $a, b \in G'$.

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[1]) The symbol $\in$ means "belongs to".

The subgroup G' of G is an invariant or normal subgroup if $hgh^{-1} \in G'$ for any $h \in G$ and any $g \in G'$.
1.3. A mapping φ of a group G_1 into another G_2 is called a homomorphism if:

$$a \to \varphi(a), \; b \to \varphi(b) \qquad \text{implies}$$

$$ab \to \varphi\,(ab) = \varphi(a)\varphi(b).$$

φ is called an *onto* homomorphism if for any $a' \in G_2$ there is an $a \in G_1$ such that $\varphi(a) = a'$.
Let e' be the unit element of G_2. The set K_φ of the elements of G_1 which are mapped into e', is called the kernel of the homomorphism φ. It is easily shown that K_φ is a normal subgroup of G_1.
A one-to-one homomorphism of G_1 onto G_2 is called an isomorphism. In this case $K_\varphi = \{e_1\}$[2]).
1.4. Let G' be a subgroup of G, and a any fixed element of G. The set of all products ah when h runs over the whole G', is called a right G' coset, indicated in what follows as aG'. If two cosets aG', bG' have one element in common they are in fact coincident. Therefore the whole group G decomposes into disjoint G' right cosets. In the same way one can define left G' cosets.
When G' is a normal subgroup of G, then for any $a \in G: aG' = G'a$. In fact the general element of aG' has the form:

$$ah \qquad h \in G'.$$

But $aha^{-1} = h', h' \in G'$; hence:

$$ah = h'a$$

that is any element of aG' is in $G'a$ and conversely. Let G' be a normal subgroup of G, and let us indicate with G/G' the set of all distinct G' cosets in G. The set G/G' equipped with the following multiplication law:

$$(aG')\,(bG') = (ab)G' \qquad a, b \in G$$

turns out to be a group, called the factor group of G with respect to G'. Due to the fact that G' is a normal subgroup, it is easy to see that the above introduced multiplication law satisfies 1.1 i-iii. We observe that the unit element of G/G' is eG'.
Given a homomorphism of a group G_1 onto a group G_2 we can form the factor group G_1/K_φ because K_φ is a normal subgroup of G_1. Observe that:

$$\varphi(a) = \varphi(b)$$

if and only if a and b belong to the same K_φ coset. In fact let e_2 be the unit element of G_2: the previous condition is equivalent to:

$$e_2 = \varphi(a)\varphi(b)^{-1} = \varphi(ab^{-1})$$

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[2]) By means of $\{a, b, c, \ldots\}$ we denote the set of the elements $a, b, c, \ldots$

i.e. $ab^{-1} \in K_\varphi$. Hence there exists $h \in K_\varphi$ such that:

$$a = hb$$

i.e. $$a \in K_\varphi b \equiv bK_\varphi.$$

Therefore we can define a mapping $\bar{\varphi}$ of G_1/K_φ onto G_2 as follows:

$$\bar{\varphi}(aK_\varphi) = \bar{\varphi}(b) \qquad b \in aK_\varphi$$

and this mapping is in fact an isomorphism of G_1/K_φ onto G_2.

1.5. Examples

a) Rotation group of the three dimensional euclidean space E_3.
We consider the set O_3 of transformationa of E_3 into itself, preserving distances and leaving unchanged a point O. Given $R_1, R_2 \in O_3$, R_1R_2 is defined as the transformation:

$$R_1R_2\boldsymbol{X} = R_1(R_2\boldsymbol{X})$$

which obviously belongs to O_3. Furthermore the identity transformation $\boldsymbol{X} \to \boldsymbol{X}$ belongs to O_3. All the O_3 transformations are one–one so that for any of them it is possible to define an inverse which of course belongs to O_3. By definition this multiplication rule is also associative, so that it gives a group structure to O_3.
We choose a set of three orthogonal axes stemming from the fixed point O. Then to any transformation:

$$\boldsymbol{X} \to \boldsymbol{X}'$$

is associated a three by three real (non singular) matrix $\{R_{ik}\}$:

$$X'_i = \sum_k R_{ik} X_k$$

satisfying the orthogonality condition:

$$RR^T = 1 \tag{1}$$

(R^T is the transpose matrix, $R^T_{ik} = R_{ki}$).
For any $R \in O_3$ the correspondence $R \to \{R_{ik}\}$ is one-to-one; in terms of the matrices, the product in O_3 reduces to the usual matrix product so that O_3 is isomorphic to this matrix group. In what follows we will identify them.
It follows from (1) that $\det R = \pm 1$. The subset of the R's with determinant $+1$, is by itself a group, called R_3, and its elements are the proper rotations of E_3. R_3 is a normal subgroup of O_3, because for any element $R \in O_3$, and for any $R' \in R_3$:

$$\det(RR'R^{-1}) = +1.$$

The factor group O_3/R_3 has only two elements: the cosets $e_1 = R_3$ and $e_2 = -I \cdot R_3$, where I is the unit matrix. Product rules are:

$$e_1e_1 = e_2e_2 = e_1$$
$$e_1e_2 = e_2e_1 = e_2.$$

b) SU_2.

The set SU_2 of two dimensional unitary unimodular (determinant = 1) complex matrices forms a group with respect to the usual multiplication law for matrices. The general form of a SU_2 matrix is:

$$U \equiv \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}; \quad |\alpha|^2 + |\beta|^2 = 1 \tag{2}$$

(the bar denotes complex conjugation).

We can express U in terms of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the unit matrix σ_0 as follows:

$$U \equiv \begin{pmatrix} \alpha_0 + i\alpha_3 & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & \alpha_0 - i\alpha_3 \end{pmatrix} = \sigma_0\alpha_0 + i \;\; \boldsymbol{\alpha} \cdot \boldsymbol{\sigma} \tag{3}$$

where $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)$ and the real numbers $\alpha_i (i = 0, \ldots, 3)$ satisfy:

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + {}_3^2 = 1 . \tag{4}$$

Putting $\quad \boldsymbol{\alpha} = -\lambda \boldsymbol{n}, \quad \boldsymbol{n}^2 = 1, \quad \lambda \geqq 0, \quad \boldsymbol{n} = -\dfrac{\boldsymbol{\alpha}}{|\boldsymbol{\alpha}|}$

by (4) we have:

$$\alpha_0^2 + \lambda^2 = 1$$

so that we can set:

$$\alpha_0 = \cos\frac{\theta}{2} \qquad 0 \leqq \theta \leqq 2\pi$$

$$\lambda = \sin\frac{\theta}{2}$$

so that:

$$U = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\, \boldsymbol{n} \cdot \sigma . \tag{5}$$

We show now that SU_2 is homomorphic to the proper rotations group R_3.

For any three dimensional vector $\boldsymbol{X}$ we define the two by two hermitian matrix:

$$\tilde{X} = \boldsymbol{X} \cdot \boldsymbol{\sigma} \tag{6}$$

and observe that:

$$\det \tilde{X} = -|\boldsymbol{X}|^2$$

For any $U \in SU_2$, we define the transformation[3]):

$$\tilde{X}' = U\tilde{X}U^+. \tag{7}$$

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[3]) If $U \equiv \{U_{ik}\}$ is a square complex matrix, we define:

$$\bar{U} : (\bar{U})_{ik} = (\overline{U_{ik}})$$

$$U^+ : (U^+)_{ik} = (\overline{U_{ki}}) .$$

The following properties are to be noted:

i) $\tilde{X}'$ is a hermitian matrix being U unitary:

ii) $\text{Tr}\,\tilde{X}' = \text{Tr}\,\tilde{X} = 0$.

it follows then: $\tilde{X}' = \boldsymbol{X}' \cdot \boldsymbol{\sigma}$ because any hermitian two by two matrix is a linear combination of the Pauli matrices with real coefficients;

iii) $\det \tilde{X}' = \det \tilde{X} = -|\boldsymbol{X}'|^2 = -|\boldsymbol{X}|^2$.

It follows that the transformation $\boldsymbol{X} \to \boldsymbol{X}'$ is a mapping of E_3 into itself which preserves the distances and does not change the origin. Hence it is a transformation of O_3. Let us indicate with $R(U)$ the element of O_3 which corresponds to the matrix U. Then from (7) we have:

$$\widetilde{R(U_1 U_2) \cdot \boldsymbol{X}} = U_1 U_2 \tilde{X} U_2^+ U_1^+ = \widetilde{U_1 R(U_2) \cdot \boldsymbol{X}} U_1^+ = \widetilde{R(U_1) R(U_2) \boldsymbol{X}}$$

i.e. $U \to R(U)$ is a homomerphism of SU_2 into O_3.

Observe that from (7) it follows that:

$$R(U) = R(-U).$$

Furthermore for any $U \in SU_2$ there exists a $V \in SU_2$ such that:

$$U = V^2. \tag{8}$$

If U is of the form (5), it is sufficient to choose:

$$V = \cos\frac{\theta}{4} - i \sin\frac{\theta}{4}\, \boldsymbol{n} \cdot \boldsymbol{\sigma}.$$

Then:

$$R(U) = R(V)\, R(V)$$

hence:

$$\det R(U) = (\det R(V))^2 = (\pm 1)^2 = 1.$$

We conclude that $U \to R(U)$ is a mapping of SU_2 into R_3 (the proper part of O_3).

Substituting expression (5) for U into (7) and carrying out the calculations, it is possible to derive an explicit expression of $\boldsymbol{X}'$ in terms of $\boldsymbol{X}$, $\boldsymbol{n}$, and θ:

$$\boldsymbol{X}' = (\boldsymbol{X} \cdot \boldsymbol{n}) \cdot \boldsymbol{n} + \cos\theta\, [\boldsymbol{X} - (\boldsymbol{n} \cdot \boldsymbol{X})\, \boldsymbol{n}] + \sin\theta\, (\boldsymbol{n} \times \boldsymbol{X}). \tag{9}$$

So that $\boldsymbol{X}'$ is obtained from $\boldsymbol{X}$ by a counterclockwise rotation of θ around $\boldsymbol{n}$.

Now let R be the rotation uniquely defined by its rotation axis (with unit vector $\boldsymbol{n}$) and rotation angle $\theta\,(0 \leq \theta < 2\pi)$ in a specified sense, to be definite in the counterclockwise sense, around $\boldsymbol{n}$: the previous formula permits us immediately to find a SU_2 matrix U sich that $R(U)$ is the given rotation:

$$U = \cos\frac{\theta}{2} - i \sin\frac{\theta}{2}\, \boldsymbol{n} \cdot \boldsymbol{\sigma}.$$

This proves that the homomorphism $U \to R(U)$ is onto. It is easily shown that the only matrices U such that

$$R(U) = 1$$

are

$$U = \pm 1$$

which consequently constitute the kernel of the homomorphism.

c) SU_3.
The set of 3×3 complex, unitary unimodular matrices also forms a group with respect to the usual matrix multiplication law, and this group is called SU_3.
d) The set of proper Lorentz transformations forms a group indicated as $L_+^{\uparrow}$ which is isomorphic to a matrix group: the set of 4×4 real matrices Λ such that:

i) $\Lambda^T G \Lambda = G$

being G the matrix:

$$G \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

ii) $\det \Lambda = 1$, $\Lambda_{44} \geq 1$

d) Inhomogeneous proper Lorentz group.

Consider the set $P_+^{\uparrow}$ of the transformations of the Minkowski space defined as:

$$X \equiv (X_1, X_2, X_3, X_4) \to X' \equiv (X_1', X_2', X_3', X_4')$$

$$X_i' = \sum_k \Lambda_{ik} X_k + a_i$$

(where $\Lambda \in L_+^{\uparrow}$, and a is a four-vector) or simply:

$$X' = \Lambda X + a .$$

Applying a transformation determined by the pair (a, Λ), and then the trans formation (a', Λ'), we obtain a new element of $P_+^{\uparrow}$

$$(a', \Lambda')\,(a, \Lambda) = (a' + \Lambda' a, \Lambda'\Lambda) .$$

With this multiplication rule $P_+^{\uparrow}$ is a group.
It is easy to see that the unit element of $P_+^{\uparrow}$ is

$$(0, I) \qquad (I = \text{unit element of } L_+^{\uparrow})$$

and that the inverse of (a, Λ) is

$$(a, \Lambda)^{-1} = (-\Lambda^{-1} a, \Lambda^{-1}) .$$

The subset

$$\{(0, \Lambda)\}$$

is a subgroup of $P_+^{\uparrow}$ isomorphic to $L_+^{\uparrow}$, and the subset

$$\{(a, I)\}$$

is an abelian subgroup, isomorphic to the translations group in four-space. Moreover $\{(a, I)\}$ is an invariant subgroup.

1.6. Topological and Lie groups

Following our definition a group is an abstract set in which there is a multiplication rule satisfying i—iii. We have subsequently checked that certain sets of matrices are groups.

It is convenient to go somewhat further and introduce in these sets a notion of nearness of two elements in such a way that the group operations enjoy a "continuity" property in a sense to be specified later. (In mathematical language this procedure is referred to as the introduction of a topology in the group).
The reasons for doing so are that many results of the group representation theory, which are especially important for physics, are based on topological properties. We will not go over the general theory of topological groups (which is comprehensively treated in many textbooks; e.g. see reference (1)); instead we will concentrate on a particular class of groups; those for which it is possible to put a one-to-one correspondence between their elements and the points of a subset of a n-dimensional real euclidean space E_n.
Let G be the group under consideration and let $\varphi(G)$ be its image in E_n. If g is an element of G and $\varphi(g)$ its image, then for any spherical neighborhood S_ε of $\varphi(g)$

$$f \in S_\varepsilon, \quad |f - \varphi(g)| < \varepsilon$$

consider the intersection of S_ε with $\varphi(G)$ (indicated as $S_\varepsilon \cap \varphi(G)$). We define as neighborhood of g in G the set Σ_ε of the elements whose image points lie in $S_\varepsilon \cap \varphi(G)$. As ε runs over real positive numbers we obtain a family of neighborhoods for each element of G and with their help one can define the concept of limit and continuity of functions on the group in the same way as in the euclidean space E_n. One can also define open sets in G: a set $S \subset G$ is open if any point of S is included in a neighborhood entirely contained in S. If the group multiplication and the inversion are continuous with respect to this topology, we will call G a topological group.
A continuous correspondence (function) between real numbers x, $0 \leq x \leq 1$, and elements $g(x)$ of a topological group G, is called a continuous path on G. The group is said to be *connected* if for any pair of elements g and g' there exists a path having them as end points.
A path $g(x)$ on G is said to be closed if $g(0) = g(1)$. Two curves $f(x)$ and $g(x)$ are said to be reconciliable when there exists a function $\Gamma(x, y)$ $(0 \leq x, y \leq 1)$ continuous in both variables, with values in G, such that:

$$\Gamma(x, 0) = f(x)$$

$$\Gamma(x, 1) = g(x).$$

In particular a closed curve $g(x)$ will be reducible to the point f if it is reconciliable with the constant function:

$$f(x) = f, \quad f \in G, \quad 0 \leq x \leq 1.$$

A group G is said to be *simply connected* if any closed curve $g(x)$ is reducible to a point.
Let us consider again the image $\varphi(G)$ of G in E_n. If $\varphi(G)$ is a compact (i.e. closed and bounded) set, then G is said to be compact. In this case any continuous real function of the elements of G is bounded (Weierstrass theorem)[4].

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[4]) It must be observed that the notion of compactness of a topological group is intrinsic, and can be given without referring to a particular parametrization of the group, just as the introduction of a topology in a group (also for this topic see [1]).

We specialize furthermore the concept of a topological group to that of a Lie group.
Let us suppose that there exists in a topological group a neighborhood N of the unit element e such that:
i) there is a one-to-one correspondence between elements of N and points of a subset of E_n. In addition we now require those parameters to be essential, i.e. it is not possible to express any of them in terms of the remaining $n-1$.
ii) if $a = a(x_1 \ldots x_n)$, $b = b(y_1 \ldots y_n)$ are elements of N such that ab and a^{-1} belong to N, and $(z_1 \ldots z_n)$, $(z'_1 \ldots z'_n)$ are respectively the parameters of ab and a^{-1}, then

$$z_i = z_i(x_1 \ldots x_n, y_1, \ldots y_n)$$

$$z'_k = z'_k(x_1 \ldots x_n)\,; \quad i, k = 1, \ldots n$$

are analytic functions of their arguments.
In this case G is said to be a n-dimensional Lie group. (In this connection a function $f(x_1, \ldots x_n)$ is said analytic at the point $(a_1, \ldots a_n)$ if there exists a neighbourhood of this point in which the function may be expressed as a converging power series of the differences $x_i - a_i$).
We will choose the parametrization of N in such a way that the set $(0, \ldots 0)$ corresponds to e.

1.7. Examples

We give here some examples to illustrate the relevant concepts introduced in the previous paragraph, as well as to establish some useful results concerning the groups which are of interest to us.
Let us begin with R_3. In sect 1.5 we have identified any rotation by a unit vector $\boldsymbol{n}$ and an angle $\theta\,(0 \leqq \theta \leqq 2\pi)$. The drawback of this is that: $R(\theta, \boldsymbol{n}) = R(2\pi - \theta, -\boldsymbol{n})$. We can instead obtain a one-to-one correspondence between rotations and three-dimensional vectors stemming from the origin of E_3, with lenght less than or equal to π: to any rotation of an angle θ $(0 \leqq \theta \leqq \pi)$ in the counter clockwise sense around a unit vector $\boldsymbol{n}$, we associate the vector $\alpha = \theta \cdot \boldsymbol{n} (|\alpha| \leqq \pi)$; conversely given α, θ and $\boldsymbol{n}$ can be obtained as:

$$\theta = |\boldsymbol{\alpha}|$$

$$\boldsymbol{n} = \frac{\boldsymbol{\alpha}}{|\boldsymbol{\alpha}|} \; (|\boldsymbol{\alpha}| \neq 0).$$

The end points of these vectors fill a sphere of radius π, and we note that the same rotation corresponds to points on the surface diametrically opposite. Hence it is necessary to identify those points in order to preserve the one-to-one correspondence property.
Since the set of parameters is a bounded closed connected subset of E_3, R_3 is a compact connected group. It is instead not simply connected, as can be seen if one considers a curve connecting two diametrically opposed points on the surface of the sphere. These two points correspond to the same element of R_3, so that the curve is effectively closed, but is cannot be reduced to a point.

Let us consider a rotation $R(\theta, \boldsymbol{n})$. From 1.5 (9) it follows that the matrix R which corresponds to the rotation R (1.5a) has the form

$$R_{ij} = (1 - \cos\theta)\, n_i n_j + \sum_{k}^{1,3} \sin\theta\, \epsilon_{ikj} n_k + \cos\theta\, \delta_{ij} \tag{10}$$

where n_k are the components of $\boldsymbol{n}$, ϵ_{ijk} is the Levi-Civita tensor, and δ_{ij} is the Kronecker tensor.
This expression is equivalent to

$$R = R(\boldsymbol{\alpha}) = e^{\theta n \cdot \Sigma} = \sum_{k}^{0,\infty} \frac{1}{k!}\, \theta^k (\boldsymbol{n} \cdot \boldsymbol{\Sigma})^k = e^{\alpha \cdot \Sigma} \tag{11}$$

where the 3×3 matrices Σ_i are defined as follows:

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \qquad \Sigma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \qquad \Sigma_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This can be seen using in (11) the relations:

$$(\boldsymbol{n} \cdot \boldsymbol{\Sigma})^2 = -1 + |n)\,(n| \tag{12}$$

$$(\boldsymbol{n} \cdot \boldsymbol{\Sigma})^3 = -(\boldsymbol{n} \cdot \boldsymbol{\Sigma}) \tag{13}$$

$$(|n)(n|)_{ij} = n_i n_j\,; \quad \alpha = \theta \boldsymbol{n}.$$

It follows then that the coefficients R_{ij} of $R(\alpha)$ are nine analytic functions of $\boldsymbol{\alpha}$, and one can verify that the Jacobian matrix

$$\left(\frac{\partial R_{ij}}{\partial \alpha_k}\right) \alpha_1 = \alpha_2 = \alpha_3 = 0$$

has characteristic 3. Hence it is possible (see COHN, [2] appendix) in a suitable neighborhood N of the point $\boldsymbol{\alpha} = (0,0,0)$, to express α_i as analytic functions of three fixed coefficients R_{ij} (say R_{12}, R_{13}, R_{23}):

$$\alpha_i = \alpha_i(R_{ij}).$$

Now consider the product $R' \cdot R'' = R$ of two elements of R_3[5]), and call respectively $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}$ the parameters of R', R'', R. The coefficients R_{ij} of R are analytic functions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Hence $\boldsymbol{\delta}$, being an analytic function of three of such coefficients, is an analytic function of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.
This demonstrates, together with the fact that the parameters of R^{-1} are obviously analytic functions of those of R, that R_3 is a 3-dimensional Lie group.
In an exactly analogous way it is possible, using (5) and the properties of Pauli matrices, to write any SU_2 matrix as:

$$U = \exp\left(-i \frac{\theta}{2}\, \boldsymbol{n} \cdot \boldsymbol{\sigma}\right) = \exp(-i \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}) \tag{14}$$

$$0 \leq \theta \leq 2\pi \quad \boldsymbol{\alpha} = \frac{\theta}{2} \cdot \boldsymbol{n}$$

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[5]) such that $R', R'', R \in N$.

so that there is a correspondence between the elements of SU_2 and the points of the sphere of radius π centered at the origin of E_3, and this correspondence is one-to-one if one identifies *all* the points of the surface with the element -1 of SU_2. Due to this fact SU_2 is not only a compact and connected group, but it is also simply connected. The same arguments as before can be used to show that SU_2 is also a 3-dimensional Lie group.

SU_3. We may identify any matrix of SU_3 with the real and imaginary part of each coefficient U_{ij}, obtaining a one-to-one correspondence between the elements of SU_3 and the points of a set of E_{18}. These parameters however are not independent, because the matrix must be unitary and unimodular. We obtain 10 conditions at all[6]), so that we have only 8 independent parameters α_K. It is possible to write any element of SU_3 in a form like (11) and (14):

$$U = e^{i\sum_K^{1,8} \alpha_K F_K} \tag{15}$$

where $F_1, F_2 \ldots F_8$ are eight hermitian traceless independent matrices that are listed, with their commutation and anticommutation rules in [*3*].

Formula (15) shows that SU_3 is a connected, 8-dimensional group and in fact it may be shown that SU_3 is also *simply*-connected.

An example of a non compact Lie group is the proper Lorentz group (1.5d). If we choose as parameters of an element of $L_+^\uparrow$ its matrix coefficients, the subset of E_{16} so obtained is not bounded; in fact, in the coefficients of matrices belonging to the subgroup of *special* Lorentz transformations there appear expressions like:

$$\frac{1}{\sqrt{1-\beta^2}} \qquad 0 \leqq \beta < 1$$

which of course are not bounded.

2. Linear Spaces

2.1. A set L of elements $x, y, z, \ldots$ is called a complex (real) linear or vector space if:

i) L is a commutative group with respect to a composition law (indicated with the symbol $+$) called sum:

$$x + y = y + x; \qquad 0 + x = x; \qquad x + (-x) = 0$$

ii) the product $\alpha x (x \in L, \alpha =$ complex (real) number, $\alpha x \in L)$ is defined so that the following conditions hold:

$$\alpha(x + y) = \alpha x + \alpha y$$
$$\alpha(\beta x) = (\alpha\beta)x = \alpha\beta x$$
$$(\alpha + \beta)x = \alpha x + \beta x$$
$$1 \cdot x = x.$$

[6]) Nine real conditions are the real and imaginary part of equations like

$$\Sigma_K u_{iK} \bar{u}_{jK} = \delta_{ij},$$

and the tenth is the condition: $\det(U_{ij}) = +1$.

2.2. n elements x_k of L are said to be linearly independent if

$$\sum_{K=1}^{n} \alpha_k x_k = 0$$

α_k = complex (real) number implies:

$$\alpha_k = 0 \qquad k = 1 \ldots n$$

otherwise they will be said linearly dependent.
We say that L is n-dimensional if there exist n linearly independent elements, whereas any $n + k$ vectors are always linearly dependent ($k \geqq 1$). Any set of n linearly independent vectors $e_i (i = 1, \ldots n)$ is called a basis for L, and we can express in a unique way any vector x as a linear combination of them:

$$x = \sum_{i=1}^{n} x_i e_i .$$

A transformation T

$$x \to x' = T(x) \qquad x \in L, \; x' \in L'$$

of the n-dimensional linear space L into the m-dimensional linear space L', is called a linear operator if

$$T(x + y) = T(x) + T(y) \qquad T(\alpha x) = \alpha T(x).$$

If $\{e_i\}$ is a basis in L and $\{e'_k\}$ is a basis in L', we have

$$x' = \sum_{k=1}^{m} x'_k e'_k = T(x) = \sum_{i=1}^{n} x_i T(e_i) = \sum_{i=1}^{n} x_i \sum_{k=1}^{m} T_{ki} e'_i$$

so that

$$x'_k = \sum_{i=1}^{n} T_{ki} x_i .$$

Hence the coordinates of the transformed vector x' are obtained from those of x, by means of a $(m \times n)$ matrix T_{ki}, that uniquely represents the given transformation in the bases $\{e_i\}$ and $\{e'_i\}$.
If in a linear space L, for any fixed n, there exist n linearly independent vectors, then L is said to be infinite dimensional.
2.3. A subset l of a linear space L, such that any linear combination of elements of l belongs to it, is called a subspace (or linear manifold) of L. L is said to be the direct sum of the subspaces $l_1, l_2, \ldots$ if it happens that any vector x of L can be expressed uniquely as a linear combination of vectors contained in $l_1, l_2 \ldots$ We will write:

$$L = l_1 \oplus l_2 \oplus l_3 \ldots$$

2.4. A correspondence of pairs of vectors of a complex linear space L into the complex numbers:

$$x, y \to (x, y)$$

satisfying the conditions:

$$(x,y) \text{ is linear in } x: \quad (\alpha z + \beta w, y) = \alpha(z,y) + \beta(w,y)$$

$$(x,y) = \overline{(y,x)}$$

$$(x,x) \geqq 0, \qquad (x,x) = 0 \quad \text{if and only if} \quad x = 0$$

is a scalar product in L.

We observe that in any finite-dimensional space it is always possible to define a scalar product (this is not true in general for infinite dimensional spaces: the existence of a scalar product must be assumed as an additional hypothesis): in fact if x_i are the coordinates of x in a fixed basis and y_i are those of y, we define

$$(x,y) = \sum_i^{1,n} x_i \overline{y}_i,$$

and it is easy to see that all the previous conditions are satisfied.

Any linear space in which a scalar product can be defined is called a Hilbert space. In the infinite dimensional case the additional hypothesis of completeness is required (see [4]).

With the aid of the scalar product it is possible to introduce the concept of lenght of a vector x:

$$\|x\| = \sqrt{(x, x)}.$$

If now T is a linear operator which maps L into itself, then we will say that T is bounded if there exists a positive number C such that for any vector $x \neq 0$:

$$\frac{\|Tx\|}{\|x\|} \leqq C.$$

To any bounded operator T it is possible to associate another one, which is called the adjoint, defined as the operator satisfying the following condition:

$$(Tx, y) = (x, T^+y)$$

$$\text{for any } x, y, \in L$$

when $T = T^+$, T is called self adjoint or hermitian; when $T^+T = TT^+ = 1$, T is said to be unitary.

3. Representation of Groups

3.1. Let G be a group and L a linear space. A representation of G in L is by definition a correspondence between elements of G and linear operators mapping L into itself, in such a way that:

$$T(g_1 g_2) = T(g_1)\, T(g_2) \qquad g_1, g_2 \in G$$

$$T(e) = I \tag{1}$$

where I is the identity operator:

$$Ix = x \qquad x \in L.$$

It follows from (1) that the operator $T(g)$ ($g \in G$) has an inverse that is $T(g^{-1})$; in fact:

$$T(g)\,T(g^{-1}) = T(gg^{-1}) = T(e) = I$$
$$T(g^{-1})\,T(g) = T(g^{-1}g) = T(e) = I.$$

The same group can be represented in finite dimensional spaces as well as in infinite dimensional spaces. In the first case we will speak of finite dimensional representation, the dimensionality of the representation being equal to that of the space.

When L is a Hilbert space, we can consider unitary representations of G, i.e. those for which $T(g)$ is a unitary operator.

3.2. Examples

Consider again R_3. We have seen that there is a one-to-one correspondence between rotation and 3×3 real orthogonal matrices with determinant equal to 1. Those matrices are operators in a 3-dimensional real linear space, and obviously this correspondence fullfills conditions (1), so that it is a representation of R_3. In addition this is a one-to-one representation, i. e. a faithful one.

Let now consider the set L^2 of the complex valued functions $\psi(\boldsymbol{x})$ defined in E_3, such that

$$\int |\psi(x)|^2\, d^3x$$

exists. This is a vector space, and also a Hilbert space, with the scalar product defined as

$$(\psi, \varphi) = \int \psi(\boldsymbol{x})\, \overline{\varphi}(\boldsymbol{x})\, d^3x.$$

The Schrödinger equation for a particle of mass m in a given potential $V(x)$ is

$$\left[\frac{\hbar^2}{2m}\nabla^2 + (E - V(\boldsymbol{x}))\right]\psi(\boldsymbol{x}) = 0 \tag{2}$$

and for certain classes of potentials (for example Coulomb potential) there exist in L^2 solutions of (2) corresponding to bound states. In this case let us call L_E the subset ($\subset L^2$) of the solutions of (2) corresponding to the same eigenvalue E. L_E is obviously a linear space.

Define for any rotation $R \in R_3$ the operator $T(R)$ in L^2 as

$$(T(R)\psi)(\boldsymbol{x}) = \psi'(\boldsymbol{x}) = \psi(R^{-1}\boldsymbol{x}).$$

If $V(x)$ is a central potential, i.e.

$$V(\boldsymbol{x}) = V(|\boldsymbol{x}|)$$

then $T(R)$ maps L_E into itself. In fact let $\psi(\boldsymbol{x}) \in L_E$ then

$$\left[\frac{\hbar^2}{2m}\nabla^2 + (E - V(\boldsymbol{x}))\right]\psi(R^{-1}\boldsymbol{x}) = \left[\frac{\hbar^2}{2m}\nabla'^2 + (E - V(\boldsymbol{x}'))\right]\psi(\boldsymbol{x}') = 0$$

where $\boldsymbol{x}' = R^{-1}\boldsymbol{x}$ and we have used the fact that

$$V(\boldsymbol{x}) = V(\boldsymbol{x}')$$

and

$$\nabla^2 = \sum_k \frac{\partial^2}{\partial x_k^2} = \sum_{i,k,j} R_{ij} R_{kj} \frac{\partial}{\partial x_i'} \frac{\partial}{\partial x_k'} = \sum_i \frac{\partial^2}{\partial x_i'^2} = \nabla'^2$$

due to orthogonality of R_{ij}.
In addition the correspondence $R \to T(R)$ satisfies (1)

$$\big(T(R_1R_2)\psi\big)(\boldsymbol{x}) = \psi(R_2^{-1}R_1^{-1}\boldsymbol{x}) = \big(T(R_2)\psi\big)(R_1^{-1}\boldsymbol{x}) = \big(T(R_1)\,T(R_2)\psi\big)(\boldsymbol{x})$$

i.e.

$$T(R_1R_2) = T(R_1)\,T(R_2)$$

and obviously the identity of R_3 is mapped into the unit operator. Hence we have a representation of R_3 in L_E, which is furthermore unitary.

3.3. a) Equivalence of representations

Let T_1 and T_2 be two representations of a given group in the spaces L_1 and L_2. They are equivalent if there exists a one-to-one linear mapping A of L_2 onto L_1 such that

$$T_1(g)\,A = A\,T_2(g)$$

for any $g \in G$. In this case we will write $T_1 \sim T_2$.
The set of all representations of G decomposes into classes of equivalent representations, and the fact that two equivalent representations are essentially the same thing, permits us to limit our study to inequivalent representations.
b) Reducible representations.
A subspace l of L is said to be invariant for a representation of G in L, if

$$T(g)\,x \in l \quad \text{when } x \in l$$

for any $g \in G$. (O and L are always invariant (trivial) subspaces). If a representation has no invariant subspaces other than O and L, it is said to be irreducible.
We observe that if $T(g)$ is a finite dimensional reducible representation of G in L, and l is an invariant subspace, we can choose a basis in L such that l is spanned by the first elements of the basis, while the remaining ones span a subspace l' ($L = l \oplus l'$).
The matrices corresponding to the operators $T(g)$ in this basis are of the block form

$$T(g) = \begin{pmatrix} T_l(g) & Q(g) \\ 0 & T_{l'}(g) \end{pmatrix} \tag{3}$$

where $T_l(g)$ maps l into itself, and $T_l(g)$ as well as $T_{l'}(g)$ define two representations of G. In fact:

$$T(g_1g_2) = \begin{pmatrix} T_l(g_1)\,T_l(g_2) & T_l(g_1)\,Q(g_2) + Q(g_1)\,T_{l'}(g_2) \\ 0 & T_{l'}(g_1)\,T_{l'}(g_2) \end{pmatrix}.$$

When $Q(g) \equiv 0$, l and l' are both invariant. In this case we will say that $T(g)$ decomposes into the direct sum of $T_l(g)$ and $T_{l'}(g)$:

$$T = T_l \oplus T_{l'}.$$

In general we will say that a representation T of G in L is decomposable if it is possible to write L as the direct sum of invariant subspaces $l_1, l_2, \ldots$ so that

$$L = l_1 \oplus l_2 \oplus \cdots = \bigoplus_i l_i$$

$$T = T_{l_1} \oplus T_{l_2} \oplus \cdots = \bigoplus_i T_{l_i}.$$

If in addition any component T_{l_i} of T is irreducible, then T is said to be completely reducible. We must observe that there are reducible representations of groups that are not decomposable. For example let us consider the set T of complex number which is a commutative group under the sum. The 2×2 matrices

$$T(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad z \in C$$

constitute a representation in the 2-dimensional complex vector space of such a group. Obviously the subspace l of vectors like

$$\begin{pmatrix} u \\ 0 \end{pmatrix}$$

is an invariant one. However no other invariant subspace exists because

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \alpha \begin{pmatrix} u \\ v \end{pmatrix}$$

implies

$$\alpha = 1, \quad zv = 0$$

which are impossible to be satisfied for any z if $v \neq 0$. Hence $T(z)$ is not decomposable.

Furthermore a decomposable representation is not always completely reducible: for example the representation

$$z \in C \quad z \to T(z) = \begin{pmatrix} 1 & z & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is decomposable but obviously not completely reducible.

However for unitary finite dimensional representations it is always true that a reducible representation is completely reducible.

It sufficies to show that if l is an invariant subspace, the orthogonal complement $l^\perp$ of l is also invariant. In fact we have:

$$g \to T(g)$$

$$g^{-1} \to T(g^{-1}) = (T(g))^{-1} = (T(g))^+$$

so that if $x \in l, y \in l^\perp$

$$T(g^{-1})\, x \in l$$

hence:

$$0 = (T(g^{-1})x, y) = (x, T(g)y)$$

i.e.

$$T(g)y \in l^{\perp} \quad \text{when} \quad y \in l^{\perp}$$

for any $g \in G$. We may then write:

$$L = l \oplus l^{\perp}$$

$$T = T_l \oplus T_{l^{\perp}}.$$

If both T_l and $T_{l^{\perp}}$ are irreducible, the theorem is proved. Otherwise there will be invariant subspaces contained in l and (or) $l_{\perp}$. In this case we repeat the above arguments, decomposing T_l and (or) $T_{l^{\perp}}$. Being L finite dimensional the process must end after a finite number of steps leading to a full reduction of T. A weaker result holds in the infinite dimensional case, namely the unitarity of the representation guarantees only its decomposability.

3.4. Characterization of the representations of compact or finite groups.

One important problem arising in the application of group theory in quantum physics is to know all the inequivalent representations of a given topological group G. This problem of the utmost importance from a purely mathematical point of view, has not been completely solved for an arbitrary topological group. However in the case of compact groups (and finite groups, i.e. those groups which contain a finite number of elements) the situation has been completely clarified by the works of Peter and Weyl, whose results we will summarize.
It is necessary in this connection to restrict our attention to those representations T satisfying the following requirements:
i) T is a continuous representation of G in a Hilbert space H; i.e. for any $g \in G$, from

$$g' \to g \quad g' \in G$$

it follows

$$||T(g')x - T(g)\,x\,|| \to 0$$

for any vector $x \in H$;
ii) if H is infinite dimensional, $T(g)$ is a bounded operator (2.4.).
Then the following statements hold:
a) in any class of equivalent representations there is a unitary representation (U.R.);
b) any irreducible representation is finite-dimensional;
c) any U.R. is completely reducible.
It sufficies then for the groups under consideration to study the finite-dimensional irreducible U.R.

4. Representations of a Compact Lie Group

4.1. In this section we want to show that in the case of Lie groups, the problem of finding out the irreducible representations is essentially equivalent to that of finding finite sets of operators obeying certain commutation rules, or, more

technically expressed, to find the irreducible representations of the Lie algebra associated to the group.
We restrict our attention to the finite dimensional representations due to the fact that we are interested in irreducible ones.
Let then $g \to T(g)$ be a finite dimensional continuous representation of the compact n-dimensional Lie group G in a vector space L. If $(\alpha_1, \ldots, \alpha_n)$ is a parametrization of a neighbourhood N of the identity e of G, then we have

$$T(g) = T(\alpha_1, \ldots, \alpha_n) \quad g \in N.$$

It can be shown [5] that the operator functions $T = T(\alpha_1, \ldots, \alpha_n)$ are analytic[7]) so that there exist n operators I_k (they will be called infinitesimal generators), defined as

$$I_k = \left(\frac{\partial T}{\partial \alpha_k}\right) \quad \begin{matrix} k = 1, \ldots n \\ \alpha_1 = \alpha_2 = \ldots = 0. \end{matrix}$$

For example if we represent R_3 with 3×3 real orthogonal matrices R_{ij} as seen before, due to the fact that

$$R = R(\theta \cdot \boldsymbol{n}) = R(\boldsymbol{\alpha}) \quad R \in R_3$$

$$T(R) = e^{\alpha \cdot \Sigma}$$

with Σ_k defined as in sect. 1.7, we have

$$I_k = \Sigma_k.$$

These operators have the commutation rules

$$[I_i, I_j] = \sum_{k=1}^{3} \epsilon_{ijk} I_k. \tag{1}$$

In the same way, from eq. (14), for the 2-dimensional representation of SU_2 we find the generators

$$I_k = -i \frac{\sigma_k}{2}$$

which satisfy commutation rules identical with (1).
4.2. We will now deduce a differential equation satisfied by the operators $T(\alpha_1, \ldots, \alpha_n)$, connecting them to the I_k [6]. Let g and f belong to G; then for any $x \in L$, we can put

$$y(g^{-1}) = T(g^{-1})x. \tag{2}$$

From 3.1 (1) it follows that

$$T(fg)\, y(g^{-1}) = T(fg)\, T(g^{-1})x = T(f)\, x = y(f)$$

i.e.

$$y(f) = T(fg)\, y(g^{-1}). \tag{3}$$

[7]) For this we mean that any matrix element $T_{ij}(\alpha_1, \ldots, \alpha_n)$ is an analytic function of $(\alpha_1, \ldots, \alpha_n)$.

Let us fix now f in N in such a way that $f^{-1} \in N$. If g is in a suitable neighbourhood of f^{-1}, then

$$g \in N$$

$$gf \in N$$

and we can write eq. (3) as:

$$y(\alpha_h(f)) = T(\alpha_i(fg))\, y(\alpha_j(g^{-1})). \tag{4}$$

Taking derivatives of (4) with respect to the parameters $\alpha_l(f)$, we obtain:

$$\frac{\partial y(\alpha_1(f), \ldots \alpha_n(f))}{\partial \alpha_l(f)} = \sum_{k=1}^{n} \frac{T(fg)}{\partial \alpha_k(fg)} \frac{\partial \alpha_k(fg)}{\partial \alpha_l(f)}\, y(g^{-1}). \tag{5}$$

It is important to note that the real functions $S_{kl} = (\partial \alpha_k(fg)/\partial \alpha_l(f))$ depend on f, g, on the group multiplication rule, on the parametrization given to N, but not on the representation $T(g)$. Letting $g \to f^{-1}$ in (5), we obtain the equation

$$\frac{\partial y(\alpha_1, \ldots \alpha_n)}{\partial \alpha_l} = \sum_{k=1}^{n} I_k S_{kl}(\alpha_1, \ldots, \alpha_n)\, y(\alpha_1, \ldots, \alpha_n) \tag{6}$$

together with the boundary condition

$$y(0, \ldots 0) = x$$

or, in an equivalent way

$$\frac{\partial T(\alpha_1, \ldots \alpha_n)}{\partial \alpha_l} = \sum_{k=1}^{n} I_k S_{kl}(\alpha_1, \ldots \alpha_n)\, T(\alpha_1, \ldots \alpha_n) \tag{7}$$

$$T(0, \ldots 0) = 1.$$

We can now demonstrate the following theorem.
If $T_1(g)$ and $T_2(g)$ are representations of a connected Lie group G in the same linear space L, and they have the same infinitesimal generators I_k, then for any $g \in G$, $T_1(g) = T_2(g)$.
In fact $T_1(g)$ and $T_2(g)$ are solutions, in a certain neighbourhood N of the unit element, of the same differential equation (7), with the same boundary condition, so that, for any element $g \in N$ we have $T_1(g) = T_2(g)$. Now in the theory of topological groups it is shown that any element g of a connected group can be expressed as a product of a finite number of elements $g_1, \ldots g_k$ belonging to an arbitrary neighbourhood of the identity: let now N be such a neighbourhood. For any $g \in G$ we have

$$g = g_1 g_2 \ldots g_k \qquad g_i \in N$$

$$T_1(g) = T_1(g_1)\, T_1(g_2) \ldots T_1(g_k) = T_2(g_1) \ldots T_2(g_k) = T_2(g)$$

which proves the theorem.

4.3. Going back to (7), we must have, for any solution $T(g)$

$$\frac{\partial^2 T(g)}{\partial \alpha_k \, \partial \alpha_l} = \frac{\partial^2 T(g)}{\partial \alpha_l \, \partial \alpha_k} \quad g \in N \tag{8}$$

(integrability conditions). For $g = e$, from (8) it follows [6]

$$[T_k, I_l] = \sum_h C^h_{kl} I_h \tag{9}$$

where the real numbers C^h_{kl} (structure constants) depend upon the derivatives of S_{kl} evaluated in $\alpha_1 = \alpha_2 = \ldots = 0$, i.e. they are independent from the particular representation chosen. By virtue of (9), C^h_{kl} satisfy

$$C^h_{kl} = -C^h_{lk}$$

$$\sum_h (C^h_{sl} C^k_{hi} + C^h_{is} C^k_{hl} + C^h_{li} C^k_{hs}) = 0. \tag{10}$$

Consider now a real vector space S of a dimension n equal to the dimension of the Lie group G, and let $\{\lambda_k\}$ $(k = 1 \ldots n)$ be a basis in S. With the aid of C^h_{kl} a composition rule in S can be defined as follows:

$$(\lambda_i, \lambda_k) \to [\lambda_i, \lambda_k] = \text{(by definition)} = \sum_k C^h_{ik} \lambda_h. \tag{11}$$

If

$$x = \sum_k x_k \lambda_k$$

and

$$y = \sum_i y_i \lambda_i \qquad x, y \in S$$

we define:

$$(x, y) \to [x, y] = \sum_{i,k} x_k y_i [\lambda_k, \lambda_i] =$$

$$= \sum_s \Big(\sum_{i,k} x_k y_i C^s_{ki}\Big) \lambda_s.$$

Due to (10), this multiplication rule has the properties

$$[x, y] = -[y, x] \qquad y, x \in S$$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \text{ (Jacobi identity)}$$

which are analogous to the usual properties of the commutator of two operators. The vector space S equipped with the composition law just defined is called the Lie algebra $\Lambda(G)$ associated to the group G. It seems that this definition depends (through C^l_{ki}) upon the particular parametrization of N. However if we make an analytic change of variables in N (i.e. if $g \in N$ and $g \equiv (\alpha'_1, \ldots \alpha'_n)$, then

$$\alpha'_i = \alpha'_i(\alpha_1, \ldots \alpha_n)$$

are invertible analytic functions in all the arguments), we obtain a set of new structure constants C'^h_{ik}, which are related to C^s_{ql} through a non singular matrix a_{ij} in the following way:

$$C'^h_{ki} = \sum_{q,l,S} a_{kq} a_{il} C^s_{ql} (a^{-1})_{sh}$$

so that it is possible to find in $\Lambda(G)$ a set of n linearly independent vectors $\lambda'_k = \sum_h a_{kh}\lambda_h$ such that

$$[\lambda'_k, \lambda'_i] = \sum_h C'^h_{ki}\, \lambda'_h .$$

We see then that to a change of parametrization in N, there corresponds in $\Lambda(G)$ only a change of basis, so that in fact $\Lambda(G)$, is uniquely determined by G. We can at this point introduce independently from the group G the concept of representation of the Lie algebra $\Lambda(G)$. By this we mean a mapping of $\Lambda(G)$ into a set of linear operators defined in a vector space L

$$x \to A(x),$$

such that
i) $A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$ (linearity)

ii) $A([x, y]) = [A(x), A(y)]$

where now $[A, B]$ means the commutator of A and B.
We then see that starting from a representation of G

$$g \to T(g)$$

its infinitesimal generators can be thought as a representation of a basis in $\Lambda(G)$, which extends by linearity to a representation of $\Lambda(G)$. The usefulness of introducing $\Lambda(G)$ is that the converse is also essentially true, in a way to be explained below.
Let then $\{\lambda_k\}$ be a basis in $\Lambda(G)$ and let

$$\lambda_k \to A_k$$

$$[A_i, A_k] = \sum_h C^h_{ik} A_h \tag{12}$$

in a finite dimensional representation of $\Lambda(G)$.
C^h_{ik} are the structure constants associated to G through a given parametrization of N. Let us consider the differential equation

$$\frac{\partial T(\alpha_1, \ldots, \alpha_n)}{\partial \alpha_k} = \sum_h S_{hk}(\alpha_1, \ldots, \alpha_n) A_h\, T(\alpha_1, \ldots, \alpha_n) \tag{13}$$

$$T(0, \ldots 0) = 1 .$$

The integrability conditions of (13), can be expressed in terms of the $S_{hk}(\alpha_1, \ldots \alpha_n)$ and it can be shown ([*1*] cap. IX) that they are satisfied in a neighbourhood of the point $(0, \ldots 0)$ due to the definition of S_{hk} (see eq. (6)) and to eq. (12), which is the form that the integrability condition assumes in the point $(0, \ldots 0)$. Hence (13) is solvable in a suitable neighbourhood N' of the point $(0, \ldots 0)$, giving us a correspondence between the elements $g \in G$, contained in a neighbourhood of the unit element, and linear operators $T'(g)$ in L.
It may be verified, in a rather cumbersome way, that if $g, g', gg' \in N'$, then

$$T(gg') = T(g)\, T(g'),$$

and for this we refer the reader to [*6*].

Let now g be an arbitrary element of G; by a result quoted previously, if G is connected, we can express g as a product of elements of N':

$$g = g_1 \cdots g_k \quad g_1, g_2, \ldots, g_k \in N'.$$

It would be tempting at this point to define

$$T(g) = T(g_1) \ldots T(g_k),$$

obtaining in this way a representation of the full group in L (it is obvious that this definition satisfies 3.1 (1)). However, if

$$g = g_1 \ldots g_k = g'_1 \ldots g'_{k'}, \quad g_i, g'_i \in N'$$

we are not sure that

$$T(g_1)\, T(g_2) \ldots T(g_k) = T(g'_1) \ldots T(g'_{k'}), \tag{14}$$

and in fact there are many cases in which they differ.
As an example of this we may consider the group G of the rotations of E_3 around the z axis. We parametrize this group with the values assumed by the rotation angle $\theta(-\pi \leqq \theta \leqq \pi)$.
To the points $\theta = \pm\pi$ it corresponds a unique element so that they must be identified. If $g(\theta_1), g(\theta_2)$ are elements of a neighbourhood N of the identity, and $g(\theta_1) \cdot g(\theta_2) \in N$, then

$$g(\theta_1)\, g(\theta_2) = g(\theta_1 + \theta_2).$$

This gives to G a structure of Lie group, and in addition G is compact and connected. This group is one dimensional and so is $\Lambda(G)$; hence any operator A on a linear space L, determines a representation of $\Lambda(G)$.
The simplest example at hand is the representation of $\Lambda(G)$ over the one dimensional complex linear space. Consider the linear operator ik (i.e. the operator that multiplies by ik, k real number). We take ik as the representative of the generator of $\Lambda(G)$, and equation (13) reads

$$\frac{\partial T(\theta)}{\partial \theta} = ikT(\theta)$$

$$T(0) = 1$$

$$T(\theta) = e^{ik\theta}$$

where θ belongs to a suitable neighbourhood N of $\theta = 0$. Let now $g(\theta)$ be an element of G. There exists an integer n such that θ/n belongs to this neighbourhood, and we can define:

$$T(g(\theta)) = T(g(\theta/n))^n = e^{ik\theta}.$$

However the element $\bar{g}$ which corresponds to $\theta = \pm\pi$ can be written (with a suitable n) as

$$g\left(\frac{\pi}{n}\right)^n \text{ and } g\left(-\frac{\pi}{n}\right)^n$$

with $g(\pi/n)$ and $g(-\pi/n)$ belonging to N; but now we have

$$T\left(g\left(\frac{\pi}{n}\right)\right)^n = e^{ik\pi} \neq T\left(g\left(-\frac{\pi}{n}\right)\right)^n = e^{-ik\pi}.$$

The failure of this method is due to the fact that G is not simply connected. For simply connected groups instead eq. (14) is fulfilled (see [7]).
In any case given a connected Lie groups G, with a Lie algebra $\Lambda(G)$, it is possible in an essentially unique way, to construct a connected Lie group G, having the same Lie algebra $\Lambda(G)$, which is in addition simply connected [1].
From this it follows the existence in $\tilde{G}$ and G of two neighbourhoods $\tilde{N}$ and N of the unit elements which are in a one-to-one bicontinuous correspondence, in such a way that if

$$\tilde{g}, \tilde{f}, \tilde{g} \cdot \tilde{f} \in N$$

$$g, f \in N$$

$$\tilde{g} \leftrightarrow g, \tilde{f} \leftrightarrow f$$

then

$$gf \in N \text{ and } \tilde{g}\tilde{f} \leftrightarrow gf.$$

This "local" isomorphism, since $\tilde{G}$ is simply connected, can be extended to a homomorphism φ of $\tilde{G}$ onto G ([1] cap. VIII). $\tilde{G}$ is called the universal covering group of G.
When G is simply connected this homomorphism reduces to an isomorphism, i.e. G and $\tilde{G}$ are essentially the same group.
At this point it should be clear that from a representation of the Lie algebra $\Lambda(G)$ of a connected Lie group G, we can construct a representation of the universal covering group of G. Let us see how to sort out from the representations of $\tilde{G}$, representations of G.
If $\tilde{g} \to T(\tilde{g})$ is a representation of $\tilde{G}$ we consider the set of those operators which correspond to the kernel K_φ of the homomorphism (see 1.3) $\tilde{G} \to G$. If this set reduces to the identity operator, then $T(\tilde{g}) = T(\tilde{f})$ when $\tilde{g}$ and $\tilde{f}$ belong to the same K_φ coset, i.e. the function $T(\tilde{g})$ is constant over each K_φ coset. Hence to any element $\tilde{g}K_\varphi$ of $\tilde{G}/K_\varphi$ it corresponds a linear operator

$$T(\tilde{g}K_\varphi) = T(\tilde{g})$$

in a way that preserves the associativity of the multiplication law in $\tilde{G}/K_\varphi$. This correspondence is then a representation of $\tilde{G}/K_\varphi$ in L and also, being $\tilde{G}/K_\varphi$ isomorphic to G, a representation of G in L.
Summarizing we can say that:
i) if G is simply connected, a representation of $\Lambda(G)$ in L determines uniquely a representation of G;
ii) if G is not simply connected, a representation of $\Lambda(G)$ in L determines a representation of $\tilde{G}$ which reduces to a representation of G if and only if the kernel K_φ of the homomorphism $\tilde{G} \to G$ is mapped into the identity operator. To exemplify let us consider again the case of R_3 and SU_2. We have seen that the

infinitesimal generators of SU_2 verify the same commutation rules as the generators of R_3, i.e. they have the same Lie algebra.
In addition SU_2 is simply connected, so that SU_2 is the covering group of R_3.
In the section 1.5b, the homomorphism of SU_2 onto R_3 has been proved, together with the fact that the kernel K_φ is the set of the two matrices ± 1. We will see later that any irreducible representation of $A(SU)_2$ is uniquely determined by an integer or half integer non negative number j, in such a way that its dimension is $2j+1$. When j is an integer the elements ± 1 are mapped into unity, so that these representations are in fact representations of R_3.
4.4. Quite independently from Lie groups, we can define a real, n-dimensional Lie algebra as a set $\mathcal{L}$ of elements such that:
i) $\mathcal{L}$ is a real n-dimensional linear space;
ii) there exists in $\mathcal{L}$ a composition law indicated as $x, y \to [x, y]$ linear in x and y, antisymmetric, satisfying Jacobi's identity (see sect. 4.3.).
As before we define a representation of $\mathcal{L}$ in a linear space L to be a mapping $x \to A(x)$ of the elements of $\mathcal{L}$ into linear operators of L satisfying conditions:

a) $A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$;

b) $A([x, y]) =$ commutator of $A(x)$ and $A(y) = [A(x), A(y)]$.

Of particular importance is the so-called regular representation of $\mathcal{L}$. In this representation $\mathcal{L}$ plays a double role in that it is the linear space in which the representation is contructed and at the same time it supplies the element to be represented

$$x \in \mathcal{L}$$

$$x \to \mathrm{ad}\,(x)$$

where $\mathrm{ad}(x)$ is the linear mapping of $\mathcal{L}$ into itself (qua vector space) defined as

$$\mathrm{ad}\,(x)y = [x, y].$$

Furthermore

$$\mathrm{ad}\,(\alpha x + \beta z) = \alpha\,\mathrm{ad}\,(x) + \beta\,\mathrm{ad}(z)$$

$$[\mathrm{ad}\,(x), \mathrm{ad}\,(z)] = \mathrm{ad}\,([x, z])$$

by virtue of Jacobi identity. Hence the correspondence

$$x \to \mathrm{ad}\,(x)$$

is in fact a representation of $\mathcal{L}$.
An invariant subspace for the regular representation is called an ideal $\mathcal{I}$:

$$y \in \mathcal{I}$$

if

$$\mathrm{ad}\,(x)y = [x, y] \in \mathcal{I} \text{ for any } x \in \mathcal{L}.$$

In particular $\mathcal{I}$ is abelian when $x \in \mathcal{I}$ $y \in \mathcal{I}$ implies $[x, y] = 0$.
Presence or absence of ideals has extremely important consequences for the structure of the Lie algebra itself and Lie algebras are divided in three classes accordingly:

i) simple Lie algebra: no ideals other than $\mathcal{L}$ and zero;
ii) semisimple Lie algebras: no abelian ideals other than zero;
iii) all the other Lie algebras.

Whereas for the first two classes there is a complete mathematical theory, the third one is not as well assessed till now.
A subalgebra of $\mathcal{L}$ is a linear subspace l such that

$$x, y \in l$$

implies

$$[x, y] \in l;$$

if in addition for any $x, y \in l$

$$[x, y] = 0,$$

we say l to be an abelian subalgebra.
Finally we mention that for any Lie algebra $\mathcal{L}$ there exists always a connected, simply connected Lie group $\tilde{G}$, such that

$$\Lambda(\tilde{G}) = \mathcal{L}.$$

Hence it is equivalent to speak of Lie algebras or of connected simply connected Lie groups. Groups related to simple (semisimple) Lie algebras, are called in turn simple (semisimple). In addition the correspondence between Lie groups and Lie algebras is such that:

there is in G		there is in $\mathcal{L}$
subgroup	$\leftrightarrow$	subalgebra
abelian subgroup	$\leftrightarrow$	abelian subalgebra
invariant subgroup	$\leftrightarrow$	ideal
abelian invariant subgroup	$\leftrightarrow$	abelian ideal

We introduce another useful concept: a subalgebra $\mathcal{C}$ is called a Cartan subalgebra, if it has the properties:
a) $\mathcal{C}$ is a maximal abelian subalgebra, i.e. there exists no other abelian subalgebra containing $\mathcal{C}$;
b) if $h \in \mathcal{C}$, then in any representation of $\mathcal{C}$ over a complex linear space, $A(h)$ is a diagonalizable operator.
For semisimple Lie algebras associated to compact Lie groups, i.e. those algebras we will use later in physical applications, one can show that any $\mathcal{L} = 0$ admits non zero Cartan subalgebras and that all Cartan subalgebras of $\mathcal{L}$ have the same dimensionality. The common dimensionality is called the rank of $\mathcal{L}$.
Finally we make a remark concerning the characterization of the connected Lie groups admitting the same Lie algebra $\mathcal{L}$. The simply connected group $\tilde{G}$ uniquely identified by $\mathcal{L}$ must be of course the covering group of all them. Hence for each of them there is a particular homomorphism $\tilde{G} \underset{\varphi}{\rightarrow} G$, so that G is isomorphic to the group $\tilde{G}/K_\varphi$. The essential feature of K_φ are:
a) K_φ is a central subgroup, i.e. its elements commute with all elements of $\tilde{G}$;
b) K_φ is a discrete subgroup, i.e. K_φ is made up with isolated elements (in $\tilde{G}$). In particular if $\tilde{G}$ is compact, K_φ has a finite number of elements.

Hence given $\tilde{G}$, one has to identify all its central discrete subgroups, which is a relatively simple task. Then by making the corresponding quotient groups, one find all the required groups.
For example see the case of R_3 and SU_2. ± 1 is the only discrete central subgroup of SU_2, so that R_3 is the only non simply connected group having the same Lie algebra as SU_2.

5. Kronecker Product of Representations

5.1. Let L and L' be two linear spaces, respectively of dimensions n and n'. Consider the set $L \otimes L'$ of all formal sums[8])

$$|x\rangle = \sum_{mm'} c_{mm'} |m\rangle\, |m'\rangle \qquad \begin{array}{l} m = 1 \ldots n \\ m' = 1 \ldots n' \end{array}$$

where $|m\rangle$ and $|m'\rangle$ are bases in L and L' and $c_{mm'}$, are arbitrary complex numbers. Defining linear combinations of elements $x, y \in L \otimes L'$ as

$$\alpha |x\rangle + \beta |y\rangle = \sum_{m,m'} (\alpha c_{mm'} + \beta b_{mm'})\, |m\rangle\, |m'\rangle$$

($b_{mm'}$ are the coefficients pertaining to y).
$L \otimes L'$ acquires a structure of linear space (of $n \cdot n'$ dimensions) and will be called the tensor or Kronecker product of L times L'.
In $L \otimes L'$ a scalar product can be defined as

$$(x, y) = \sum_{m,m'} c_{mm'}\, \bar{b}_{mm'}\,.$$

If now $g \to T(g)$ and $g \to T'(g)$ are two representations of the same group G in L and L', a new representation of G in $L \otimes L'$ (indicated as $T \otimes T'$) can be defined as follows:

$$g \to T(g) \otimes T'(g)$$

where

$$(T(g) \otimes T'(g))\, |m\rangle\, |m'\rangle = \sum_{ss'} T_{sm}(g)\, T'_{s'm'}(g)\, |s\rangle\, |s'\rangle$$

and for an arbitrary vector $|x\rangle$:

$$\begin{aligned}(T(g) \otimes T'(g))\, |x\rangle &= \sum_{mm'} c_{mm'} (T(g) \otimes T'(g))\, |m\rangle\, |m'\rangle = \\ &= \sum_{ss'} \Big(\sum_{mm'} c_{mm'}\, T_{ms}(g)\, T'_{m's'}(g) \Big)\, |s\rangle\, |s'\rangle .\end{aligned}$$

Hence we see that each element x of $L \otimes L'$ is uniquely determined by a set of $n \cdot n'$ complex numbers $c_{mm'}$, and under the representation $T \otimes T'$ these components transform as

$$c'_{mm'} = \sum_{ee'} T_{em}\, T'_{e'm'}\, c_{ee'}\,.$$

[8]) Where it is convenient we use for vectors the Dirac notation: $|m\rangle$ for e_m (sect. 2.2).

It is easy to see that $T \otimes T'$ is unitary when T and T' are; but even in the case that T and T' are irreducible, $T \otimes T'$ is not so. However, according to the general statements in section (3.4b—c) being $T \otimes T'$ unitary, it will be fully reducible: the space $L \otimes L'$ can be written as a direct sum of invariant subspaces $l_{i,\alpha}$ which transform according to irreducible representations $T_{i,\alpha}(g)$[9]):

$$L \otimes L' = \bigoplus_{i,\alpha} l_{i,\alpha}$$

$$T \otimes T' = \bigoplus_{i,\alpha} T_{i,\alpha} \qquad i = 1 \ldots K.$$

Let us choose in each $l_{i,\alpha}$ an orthonormal basis of vectors indicated as

$$|r; i, \alpha\rangle$$

where the index r labels the basis vectors of $l_{i,\alpha}$. Collecting all these vectors we obtain an orthonormal basis in $L \otimes L$; hence there exists a unitary matrix

$$C(m, m'; r, i, \alpha)$$

connecting-this basis to that previously introduced

$$|m\rangle\,|m'\rangle = \sum_{r,i,\alpha} C(m, m'; r, i, \alpha)\,|r; i. \alpha\rangle$$

$$|r; i, \alpha) = \sum_{mm'} \overline{C}(m, m'; r, i, \alpha)\,|m\rangle\,|m'\rangle.$$

The quantities $C(m, m'; r, i, \alpha)$ are called Clebsch-Gordan coefficients of the group G.
We observe that due the fact that $l_{i\alpha}$ and $l_{i\beta}$ transform according to equivalent representations (sect. 3.3a), their elements can be put in a linear one-to-one correspondence, so that we can choose vectors $|r, i\alpha\rangle$, $|s, i\,\beta\rangle$ in such a way that indices r and s run over the same range. In addition these vectors can be choosen so to satisfy:

$$\langle i\alpha r'|T(g)|i\alpha r\rangle = \langle i\alpha r'|T_{i\alpha}(g)|i\alpha r\rangle = \langle i\beta r'|T_{i\beta}(g)|i\beta r\rangle = \langle i\beta r'|T(g)|i\beta r\rangle.$$

We will always refer to a basis selected in this way whenever we will have to deal with reducible representations of a group G, calling it the standard basis.

6. Schur's Lemma and Wigner-Eckart Theorem

6.1. Schur's lemma

i) Let $T_1(g)$ and $T_2(g)$ be two irreducible, inequivalent, finite dimensional representations of a group G in the linear spaces L_1 and L_2.

[9]) In general there will be several irreducible subspaces transforming according the same irreducible representation. These subspaces are distinguished by the additional label α, whereas i distinguishes between group of subspaces transforming according inequivalent representations.

Any linear operator A mapping L_1 into L_2, such that

$$T_2(g)\,A = A\,T_1(g) \quad \text{for any} \quad y \in G$$

is the null operator.
Proof: Let N_A be the set of vectors in L_1 such that

$$Ax = 0 \quad x \in N_A.$$

N_A is an invariant subspace for $T_1(g)$. In fact if $x \in N_A$, then

$$A\,T_1(g)x = T_2(g)Ax = 0$$

i.e.

$$T_1(g)x \in N_A \quad \text{for any} \quad g \in G.$$

Then $N_A = L_1$ or $N_A = 0$. In the first case the theorem is proved. In the second case let us call R_A the image of L_1 into L_2. R_A is invariant for $T_2(g)$:

$$y \in R_A$$

i.e.

$$y = Ax$$

$$T_2(g)y = T_2(g)Ax = A\,T_1(g)x$$

i.e.

$$T_2(g)y \in R_A \quad \text{when} \quad y \in R_A.$$

Hence $R_A = 0$ or $R_A = L_2$. The second case is excluded being T_1 and T_2 inequivalent, which proves the theorem.
This theorem can be extended to infinite dimensional representations, provided A is a bounded operator.
ii) If T_1 and $T_2(g)$ are irreducible, equivalent, finite dimensional representations of G in the complex vector spaces L_1 and L_2, i.e. there exists a one-to-one mapping U of L_1 into L_2 such that $T_2 U = U T_1$, then any linear operator A mapping L_1 into L_2 and satisfying

$$T_1 A = A\,T_2$$

is a multiple of U: $A = \lambda U$.
Proof: From

$$T_2(g)A = A\,T_1(g) \quad \text{and}$$

$$U^{-1}T_2(g)\,U = T_1(g) \quad \text{it follows}$$

$$T_2(g)\,A = A\,U^{-1}T_2(g)\,U$$

i.e.

$$T_2(g)A\,U^{-1} = A\,U^{-1}T_2(g)$$

so that the theorem is proved if we show that any operator A' which commutes with all the operators of an irreducible representation of G is a multiple of the unit element (in fact if this is true we have $A\,U^{-1} = \lambda I$ i.e. $A = \lambda U$).
Any operator A' has at least an eigenvector $x \neq 0$ belonging to some eigenvalue λ:

$$A'x = \lambda x.$$

Let V be the linear manifold spanned by the vectors belonging to this eigenvalue $(V \neq 0)$. V is invariant under $T_2(g)$ in fact:

$$x \in V$$

implies

$$A' T_2(g) x = T_2(g) A' x = \lambda T_2(g) x$$

i.e.

$$T_2(g) x \in V .$$

But $V \neq 0$, and T_2 is irreducible. Hence it follows $V = L_2$, i.e.

$$A' = \lambda I .$$

As before, the theorem is true for any bounded operator A' commuting with the operators $T(g)$ of an unitary irreducible representation of G in any Hilbert space.
6.2. We use the results of sect. 6.1 to determine the structure of an operator T, which is invariant under an arbitrary unitary representation of a group G.
In particular we will determine the form of its matrix elements, with respect to a fixed basis.
To be definite let $g \to U(g)$ be an unitary representation of G in a Hilbert space $\mathscr{H}$. We require it to be completely reducible, which is always the case when G is compact or finite. Let T be an operator mapping $\mathscr{H}$ into itself such that:

$$T U(g) = U(g) T \quad \text{for any} \quad g \in G . \tag{1}$$

$\mathscr{H}$ can be decomposed (sect. 5.1) into a direct sum of invariant irreducible subspaces:

$$\mathscr{H} = \bigoplus_{\alpha} (\bigoplus_{i} l_{i\alpha}) = \bigoplus_{i\alpha} l_{i\alpha} . \tag{2}$$

For any vector $\Phi_{i\alpha} \in l_{i\alpha}$, we have

$$U(g)\, \Phi_{i\alpha} \in l_{i\alpha}$$

so that we may define an operator $U^{i\alpha}(g)$ mapping $l_{i\alpha}$ into itself as

$$U^{(i\alpha)}(g)\, \Phi_{i\alpha} = U(g)\, \Phi_{i\alpha} .$$

$U^{(i\alpha)}(g)$, called the restriction of $U(g)$ to $l_{i\alpha}$, by hypothesis constitute an irreducible representation of G specified by the labels $i\alpha$

$$\big(U^{(i\alpha)}(g) \sim U^{(i\beta)}(g)\big) .$$

Consider now the vector $T\Phi_{j\alpha}$ $(\Phi_{i\alpha} \in l_{i\alpha})$. By (2) we can uniquely write

$$T\, \Phi_{i\alpha} = \sum_{j\beta} \psi_{j\beta} \; (\psi_{j\beta} \in l_{j\beta}) \tag{3}$$

and define the operators $T^{(i\alpha)}_{(j\beta)}$, mapping $l_{i\alpha}$ into $l_{j\beta}$ as

$$T^{(i\alpha)}_{(j\beta)} \Phi_{i\alpha} = \psi_{j\beta} ; \quad T\, \Phi_{i\alpha} = \sum_{j\beta} T^{(i\alpha)}_{(j\beta)} \Phi_{i\alpha} . \tag{4}$$

We use now (1), which gives:

$$U T \Phi_{i\alpha} = \sum_{j\beta} \psi U_{j\beta} = \sum_{i\beta} U^{(j\beta)} \psi_{j\beta} = \sum_{j\beta} U^{(j\beta)} T^{(i\alpha)}_{(j\beta)} \Phi_{i\alpha} = \sum_{j\beta} \psi'_{j\beta} =$$

$$= T U \Phi_{i\alpha} = T U^{(i\alpha)} \Phi_{i\alpha} = \sum_{j\beta} T^{(i\alpha)}_{(j\beta)} U^{(i\alpha)} \Phi_{i\alpha} = \sum_{j\beta} \psi''_{j\beta}.$$

By definition of direct sum, we then obtain:

$$\psi'_{j\beta} = \psi''_{j\beta}$$

i.e.

$$U^{(j\beta)} T^{(i\alpha)}_{(j\beta)} = T^{(i\alpha)}_{(j\beta)} U^{(i\alpha)}. \tag{5}$$

$U^{(i\alpha)}$, $U^{(j\beta)}$ are irreducible representations of G on $l_{i\alpha}$ and $l_{j\beta}$, so that we can apply Schur's lemma to the operator $T^{(i\alpha)}_{(j\beta)}$ concluding that:

a) $T^{(i\alpha)}_{(j\beta)} \equiv 0$ when $i \neq j$

b) $T^{(i\alpha)}_{(i\beta)} = \lambda(i\alpha\beta)\, V^{i\alpha}_{i\beta}$ where $\lambda(i\alpha\beta)$ depends upon i, α, β and $V^{i\alpha}_{i\beta}$ is a fixed operator mapping $l_{i\alpha}$ into $l_{i\beta}$, such that

$$U^{i\beta} V^{i\alpha}_{i\beta} = V^{i\alpha}_{i\beta} U^{i\alpha}. \tag{6}$$

In particular, choosing inside each $l_{i\alpha}$ a standard basis $\{\Phi^{i\alpha}_r\}$ as in sect. 5.1, the operator defined as

$$V^{i\alpha}_{i\beta} \Phi^{i\alpha}_r = \Phi^{i\beta}_r$$

satisfies (6).
In conclusion, using a), b), (3) we see that:

$$(\Phi^{j\beta}_s, T\Phi^{i\alpha}_r) = \lambda(i, \alpha\beta)\, \delta_{ij} \delta_{rs}. \tag{7}$$

6.3 Wigner-Eckart theorem

This theorem, which is valid for any compact group, can be seen as a generalization of the preceeding statements on matrix elements of invariant operators. Let $g \to U(g)$ be an unitary representation of G into the Hilbert space $\mathscr{H}$. Suppose we have a finite number of operators T_k such that

$$U(g)\, T_k\, U(g^{-1}) = (D^j(g))_{k'k} T_{k'}$$

where $D^j(g)$ is a matrix of the irreducible representation of G labeled by j. Operators of this kind are called irreducible tensor operators transforming as the representation j.
Consider now a decomposition of $\mathscr{H}$ into irreducible subspaces $l_{i\alpha}$, and a standard basis $\{\Phi^{i\alpha}_r\}$.
Wigner-Eckart theorem states:
Given a set of irreducible operators T^l_k transforming as the representation of G specified by l, then:

i) the matrix element:

$$(T^l_k \Phi^{i\alpha}_r, \Phi^{j\beta}_s) \tag{8}$$

vanishes whenever the representation j is not contained in the Kronecker product of the representations l and i.
ii) when the representation j is contained in this tensor product, then:

$$(T_k^l \Phi_r^{i\alpha}, \Phi_s^{j\beta}) = \sum_\gamma C(i\,r, l\,k; j\,s\,\gamma) \langle i\,\alpha || T^l || j\,\beta \rangle_\gamma \tag{9}$$

where $C(ir, lk; js\gamma)$ are Clebsch-Gordan coefficients which project the vector $\Phi_r^i \Phi_k^l$ of the tensor product of the representations i and l into the vector $\Phi_s^{j\gamma}$ transforming as the γ^{th} irrducible component equivalent to the representation j. $\langle i\alpha || T^l || j\beta\rangle_\gamma$ is a symbolic way of writing a number which depends no more upon the "magnetic" quantum numbers k, r, s, and is called a reduced matrix element of T_k^l. In this way the dependence of (8) upon the magnetic quantum numbers is lumped into Clebsch-Gordan coefficients, i.e. is the same for any set of irreducible operators transforming in a fixed way, and here is the main importance of the theorem.
The number of terms appearing in (9) is simply the number of times the representation j appears in the tensor product of representations i and l.
6.4. We want to visualize the important results obtained in last two sections with a simple example.
Let us consider a representation of SU_2 which is the direct sum of two irreducible representations of $j = 1, 1/2$.
In this case:

$$H = l_1 \oplus l_{1/2} \qquad \text{(no need for any index like } \alpha)$$
$$U(g) = U^1(g) \oplus U^{1/2}(g).$$

A standard basis is one in which the third generator of SU_2 is diagonal: vectors of this basis will be indicated as

$$|j, m_j\rangle \qquad j = 1, {}^1/_2$$
$$m_j = \begin{cases} +1,\ 0,\ -1 & j = 1 \\ {}^1/_2, \quad -{}^1/_2 & j = {}^1/_2. \end{cases}$$

Consider a set of two operators $T_i^{1/2} (i = \pm {}^1/_2)$ transforming as the $j = {}^1/_2$ representation, i.e.

$$U(g)\, T_i^{1/2}\, U(g)^{-1} = \sum_k (U^{1/2}(g))_{ki}\, T_k^{1/2}, \quad g \in SU_2.$$

Let us find the structure of matrix elements of $T_i^{1/2}$, using Wigner-Eckart theorem. From (35) we have

$$\langle j', m'_{j'} \mid T_i^{1/2} \mid j, m_j\rangle = C_{m_j\, i\, m'_{j'}}^{j\, {}^1/_2\, j'}\, \lambda(j, j').$$

The decomposition of tensor products involved here is as follows:

$$U^{1/2} \otimes U^{1/2} = U^1 \oplus U^0$$
$$U^1 \otimes U^{1/2} = U^{3/2} \oplus U^{1/2}.$$

Hence when $j = 1$ the matrix elements vanish unless $j' = {}^1/_2$ and viceversa. Non vanishing matrix elements are:

$$\langle 1\, m' \,|T_i^{1/2}|\, {}^1/_2 m\rangle = C^{1/2\,1/2\,1}_{m\,i\,m'}\,\lambda(1, {}^1/_2)$$

$$\langle {}^1/_2\, m' \,|T_i^{1/2}|\, 1\, m\rangle = C^{1\,1/2\,1/2}_{m\,j\,m'}\,\lambda({}^1/_2, 1).$$

Using Clebsch-Gordan coefficients of Condon and Shortley we find the following structure:

$$T^{1/2}_{1/2}:\quad \begin{array}{c} \\ \\ \end{array}\left(\begin{array}{ccccc} 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \frac{\lambda}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}}\,\mu & 0 & 0 \end{array}\right) \begin{array}{cc} m & j \\ 1 & \\ 0 & 1 \\ -1 & \\ {}^1/_2 & \\ -{}^1/_2 & {}^1/_2 \end{array}$$

$$m:\ \underbrace{1\ \ 0\ \ -1}_{j:\ 1}\quad \underbrace{{}^1/_2\ \ -{}^1/_2}_{{}^1/_2}$$

$$T^{1/2}_{-1/2}:\quad \begin{array}{cc} 1 & 1 \\ & 0 \\ & -1 \\ {}^1/_2 & {}^1/_2 \\ & -{}^1/_2 \end{array}\left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \lambda \\ \sqrt{\frac{2}{3}}\,\mu & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{\sqrt{3}} & 0 & 0 & 0 \end{array}\right)$$

$$\underbrace{1\ \ 0\ \ -1}_{1}\quad \underbrace{{}^1/_2\ \ -{}^1/_2}_{{}^1/_2}$$

(we set $\lambda(1\,{}^1/_2) = \lambda$, $\lambda({}^1/_2, 1) = \mu$).

Finally we consider the case of an invariant operator, i.e. an operator T such that:

$$U(g)\, T\, U(g^{-1}) = T.$$

In this case, using formulas of sect. 6.2, we find the following structure:

$$\begin{array}{cc} 1 & 1 \\ & 0 \\ & -1 \\ {}^1/_2 & {}^1/_2 \\ & -{}^1/_2 \end{array}\left(\begin{array}{ccc|cc} \lambda^1 & 0 & 0 & & \\ 0 & \lambda^1 & 0 & & 0 \\ 0 & 0 & \lambda^1 & & \\ \hline & 0 & & \lambda^{1/2} & 0 \\ & & & 0 & \lambda^{1/2} \end{array}\right)$$

$$\underbrace{1\ \ 0\ \ -1}_{1}\quad \underbrace{{}^1/_2\ \ -{}^1/_2}_{{}^1/_2}$$

7. Symmetry Principles in Elementary Particles Physics

7.1. In the development of Quantum Physics increasing attention has been given, at various stages, to invariance principles. Particularly, in the physics of elementary particles the systematic inquiry and exploitation of the so called internal

symmetries led, in the last few years, to remarkable successes in contrast to the unsatisfactory status of dynamical calculations. It appears that, due to peculiar features of the quantum mechanical description, the theory of group representations provides the natural device to handle such invariance principles.
Physical laws express correlations among observable events. The latter of course take place in space and time so that to specify any of them we need certain coordinates with respect to a fixed space-time frame of reference. In addition we need some other "internal parameters" (such as charge, baryonic number etc.) which uniquely determine the nature of the objects under consideration.
Physical laws are then functional relations among sets of such "coordinates". Loosely speaking a symmetry is a transformation on the coordinates which leaves invariant these relations.
There are invariance principles which we believe to be valid for any kind of phenomena, and these are: invariance under translations in space and time and spatial rotations. As stressed by WIGNER [8], the mere possibility of comparing results of experiments made in different places and at different times (i.e. the reproducibility of phenomena) is based on this assumption.
At the same level of universality we accept the assumption that the physical laws are the same in all frames of reference differing for an uniform rectilinear motion. All this is summarized in the statement that Physics is invariant under inhomogeneous proper Lorentz transformations.
There are many other symmetries in elementary particle physics which are shared only by certain kinds of processes (as for example is the case for the SU_2 or isotopic spin symmetry which is valid only for strong interactions). We postpone the study of these topics to a brief sketch of Wigner's analysis of relativistic invariance in quantum mechanics.
7.2. In the formalism of quantum mechanics, there is a normalized vector ψ in a Hilbert space $\mathscr{H}$, corresponding to any physical situation we can set up in laboratory. The normalization condition determines ψ only up to a phase factor, so that what is really relevant is a set Ψ of vectors different from one another by phase factors. Ψ is called a unit in $\mathscr{H}$. There is in addition a self-adjoint operator A corresponding to any observable quantity a and the connection between theory and experiments is contained in the statement that the average value of a in the situation Ψ is

$$m_{\Psi}(a) = (A\psi, \psi)$$

where ψ is an arbitrary vector of the ray Ψ. As it is well known, any physical quantity can be written in terms of expressions like:

$$|(\psi, \varphi)|^2$$

and these depend only upon the rays Ψ, Φ to which ψ, φ belong.
It is an experimental fact that physically realizable states always correspond to definite values of charge, baryonic number (N) and leptonic number (l). This has the consequence that a vector in $\mathscr{H}$ which is a superposition of two states with different eigenvalues of Q or N or l cannot correspond to a physically reali-

zable situation. Hence $\mathscr{H}$ breaks up into subspaces called coherent sectors such that the superposition principle holds only within each coherent sector.
Such phenomenon is the manifestation of the so-called superselection rules [9, 10].
Consider now a system prepared in a certain state Ψ. If g is an element of the proper inhomogeneous Lorentz group $P_+^{\uparrow}$ we can apply this transformation to the instruments with which we have prepared Ψ, obtaining a new physical situation of the same system, described by a ray Ψ^g. As stated by Wigner, the theory is relativistically invariant if the following requirements are satisfied.
i) The mapping $\Psi \to \Psi^g$ is one-to-one, and maps each coherent sector into a coherent sector;
ii) if $\Psi \to \Psi^g$, $\Phi \to \Phi^g$, then:

$$|(\psi, \varphi)| = |(\psi^g, \varphi^g)|$$

where ψ, φ, ψ^g, φ^g are vectors belonging to Ψ, Φ, Ψ^g, Φ^g;
iii) $(\Psi^g)^{g'} = \Psi^{(gg')}$ and

$$\text{when } g' \to g \quad \Psi^{g'} \to \Psi^g.$$

The existence of Ψ^g for any Ψ and g is equivalent to what is called the homogeneity of Minkowski space-time, and combined with i) makes us sure that all physical operations possible in a given frame, are possible in any other frame connected to it by a Lorentz transformation. Condition ii) tells us that the connections between any two states depend only upon their relative motion or position. Finally iii) merely expressed the fact that $P_+^{\uparrow}$ is a group and that two slightly different transformations must produce nearly the same effect. From i) and iii) it follows that Ψ and Ψ^g belong to the same coherent sector.
From i)—ii) it can be shown [11, 12] that there is a unitary operator $U(g)$ corresponding to each $g \in P_+^{\uparrow}$, such that

$$U(g)\psi \in \Psi^g \quad \text{when} \quad \psi \in \Psi.$$

$U(g)$ is defined up to a factor of modulus one, in that the substitution $U(g) \to U'(g) = \omega(g)\, U(g)$ $(|\omega(g)| = 1)$ gives us another set of admissible operators.
However there exist a neighborhood N of the unit element in $P_+^{\uparrow}$ and a particular choice of these phase factors such that

a) $g \in N$, $g \to U(g)$ is a continuous mapping;

b) $g, g', g \cdot g' \in N$ implies $g \cdot g' \to U(gg') = U(g)\, U(g')$,

so that we have a local representation of $P_+^{\uparrow}$ in $\mathscr{H}$. Being $P_+^{\uparrow}$ not simply connected this local representation extends to a representation of the covering group $\widetilde{P_+^{\uparrow}}$. Its infinitesimal generators, multiplied by $-i$, are ten self adjoint operators P_μ and $M_{\mu\nu} = -M_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$, P_μ being identified with the total momentum (so that P_0 is the total hamiltonian), whereas

$$J^i = \sum_{h,k} \varepsilon^{ihk} M_{hk}, \quad h,k = 1,2,3$$

are the total angular momentum operators of the system. P_μ and $M_{\mu\nu}$ are a representation of the Lie algebra of $P_+^{\uparrow}$. Their commutation relations are listed in [9]. In particular

$$[P_\mu, P_0] = [J^i, P_0] = 0$$

so that momentum and angular momentum conservation follows from Lorentz invariance.
From previous considerations we have seen that the only possible relativistically invariant descriptions of a quantum system are given in terms of unitary representation of $\widetilde{P_+^{\uparrow}}$ in the Hilbert space of state vectors. When these representations are irreducible we speak of elementary system. In this case any state can be reached from a fixed one by means of a Lorentz transformation, and there is no way to divide the Hilbert space into subsets transforming independently under $\widetilde{P_+^{\uparrow}}$.
The study of the irreducible unitary representations of $\widetilde{P_+^{\uparrow}}$ has been carried out by WIGNER [12] and the results are as follows: each irreducible representation is characterized by two numbers m and s. m^2 is the eigenvalue of $\sum_\mu P^\mu P_\mu$ in the representation[10]) and is, according to our previous identification of P_μ with the total momentum, the mass squared of the particle. Hence the only cases of interest for physics are the irreducible representations with $m^2 > 0$ or $m^2 = 0$. In the first case s is an integer or half integer non negative number, and is equal to the spin of the particle. In the second case $s = 0, \pm {}^1/_2, \pm 1, \ldots$ and is the component of the particle spin along the direction of motion (helicity).
A very detailed analysis of Lorentz group representations as well as their application to scattering processes can be found in [13].
We conclude emphasizing that from the previous considerations we have extracted a very precise definition of elementary particle in a quantum theory, at least as far as its space-time behaviour is concerned: it is a system whose states transform like an irreducible representation of $\widetilde{P_+^{\uparrow}}$, and it is thus uniquely determined by its mass and spin.
7.3. Apart from space-time symmetries, some kind of interactions between particles exhibit peculiar invariance properties: in particular we will focus our attention on symmetries of strong interactions.
Should we know the actual dynamical structure of strong interactions then it would be possible to check directly what are the transformation on internal labels which leave such dynamics invariant (in the sense of sect. 7.1).
Let us see in a particular model (field theory) what type of conclusions can be drawn from the existence of a Lie group of such transformations.
This will be a guide for us in the actual situation where we do not know the dynamics involved, in order to be able to guess, from certain experimental observations, the existence of strong interactions symmetries.
Suppose that strongly interacting particles are described by certain fields $\psi_\alpha(x)$ and by a Lagrangian $\mathcal{L}(\psi_\alpha, \partial_\mu\psi_\alpha)$.
Moreover suppose that there exists a certain n-dimensional Lie group G of transformations on the fields

$$\psi_\alpha(x) \to \psi'_\alpha(x) = U\psi_\alpha(x)U^{-1} = \sum_\beta t_{\alpha\beta}\psi_\beta(x) \tag{1}$$

which leave $\mathcal{L}$ invariant (here the labels α,β refer only to internal degrees of freedom such as charge, baryonic number, hypercharge etc. whereas space-time labels are neglected).

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[10]) Due to the fact that $\sum_\mu P_\mu P^\mu$ commutes with all the infinitesimal generators, in any irreducible representation is must be a multiple of the unit operator.

Such transformations are induced on the fields by unitary operators $U(g)$ ($g \in G$) which are a representation of G in the Hilbert space of the strongly interacting particles. The infinitesimal generators multiplied by $-i$ are n selfadjoint operators Q_k. Their expression in terms of the fields can be found by requiring that the infinitesimal transformations leave unchanged the Lagrangian. In fact one can construct in terms of the fields n divergence-free currents

$$J_\mu^k(x) \qquad k = 1, \ldots n$$

$$\partial^\mu J_\mu^k(x) = 0 \tag{2}$$

such that the operators

$$\int J_4^k(x)\, d^3x$$

satisfy the commutation relations characterizing the Lie algebra of G, and are just the infinitesimal generators of the transformation (1). From (2) it follows that such operators are constant of motion (this result is just the quantum counterpart of the classical Noether's theorem [*14*]).

Consider now one particle states. They are obtained by applying to the vacuum state a creation operator a_α^+ (again space-time labels are omitted) which obviously satisfies

$$U a_\alpha^+ U^{-1} = \sum_\beta t_{\alpha\beta} a_\beta^+$$

so that

$$U\,(a_\alpha^+ \mid 0\rangle) = \sum_\beta t_{\alpha\beta}\,(a_\beta^+ \mid 0\rangle)$$

i.e. one particle states transform like a representation of G. Moreover, since the Q_k's are constant operators, the Hamiltonian which is the time displacements generator, commutes with them, being therefore an invariant operator under G; the same applies to the mass operator. If G is compact, then the representation of G over one particle states is completely reducible and the irreducible components correspond to states of particles with the same mass.

We can simultaneously diagonalize a number of Q_k equal to the rank of G, then we find multiplets of particles with equal masses, distinguished (apart from possible degeneracies[11]) by the eigenvalues of the diagonal Q_k's, i.e. by certain "internal" quantum numbers.

Passing to multiparticle states we observe that they transform under G as follows

$$U(a_{1\alpha}^+ a_{2\beta}^+ \ldots \mid 0\rangle) = U a_{1\alpha}^+ U^{-1}\, U a_{2\beta}^+ U^{-1} \ldots \mid 0\rangle =$$
$$= \sum_{\alpha',\beta'} t_{\alpha\alpha'},\, t_{\beta\beta'} \ldots (a_{1\alpha'}^+,\, a_{2\beta'}^+ \ldots \mid 0\rangle)$$

(1, 2 . . . take into account space time as well as other degrees of freedom which are uneffected by G) i.e. as tensor product of one particle representations.

Suppose

$$\mid q_1^k, 1\rangle;\ \mid q_2^k, 2\rangle; \ldots$$

to be one particle states which are eigenstates of Q_k with eigenvalues $q_1^k, q_2^k, \ldots$ Then, under the unitary transformation $U = (1 + i\varepsilon Q_k)$, the multiparticle

[11]) This happens e.g. in the case of SU_3.

state

$$| q_1^k, 1\rangle \, | q_2^k, 2\rangle \ldots \tag{4}$$

transforms as

$$(1 + i\varepsilon Q_k) \, | q_1^k, 1\rangle \, | q_2^k, 2\rangle \cdots = (1 + i\varepsilon Q_k) \, | q_1^k, 1\rangle \, (1 + i\varepsilon Q_k) \, | q_2^k, 2\rangle \cdots =$$

$$= | q_1^k, 1\rangle \, | q_2^k, 2\rangle \cdots + i\varepsilon (q_1^k + q_2^k + \cdots) \, | q_1^k, 1\rangle \, | q_2^k, 2\rangle \cdots + O(\varepsilon^2)$$

and we conclude that the state (4) is an eigenstate of Q_k with eigenvalue $q_1^k + q_2^k + + \cdots$, i.e. the Q_k's are *additive conserved quantities*.

In conclusion: starting from invariance under an n-dimensional Lie group G, we have found:

a) n additive conserved quantities (in general not all simultaneously diagonalizable);

b) a multiplet structure for the one particle states;

c) consider furthermore the scattering of two particles into an arbitrary multiparticle state. The corresponding amplitudes are given by the matrix elements of the S-operator which, being a function of the Lagrangian $\mathcal{L}$, turns out to be an invariant operator under G, as $\mathcal{L}$ is. We already saw in sect. 6.2 the general structure of matrix elements of such an operator between states belonging to arbitrary representations of the group G. That analysis tells us that symmetry under G severely restricts the form of the S-matrix, leading to relations between amplitudes of a-priori uncorrelated processes.

At this point one remark is in order. It may well be (and this is the case for isotopic spin or SU_2 symmetry) that not all the Q_k's commute with charge, or with some among the other observables which define a superselection rule. When this is the case these Q_k's do not have a complete set of observable states and their conservation cannot be directly observed. If the symmetry has to be useful, at least a number equal to the rank of G among the Q_k's has to sommute with each observable defining a super selction rule (as well as among themselves), in order to use their common eigenvalues as labels for physical states. Point a) is then reduced to the existence of at least r additive mutually commuting conserved quantities (r = rank of G).

A simple example of an invariance principle which can be treated in this way is the so called first kind gauge invariance.

Suppose the following transformations to leave unchanged the Lagrangian: for any Hermitian field φ: $\varphi \to \varphi$

for any non Hermitian field ψ corresponding to $+1$ charged particles: $\psi \to e^{i\alpha}\psi$

for the adjoint field $\overline{\psi}$: $\overline{\psi} \to e^{-i\alpha}\overline{\psi}$ (α real).

They constitute a one-dimensional Abelian Lie group[12]).

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[12]) This group called U_1 has as elements the complex number $e^{i\alpha}$ (α mod 2π), with the multiplication law:

$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$$

As for any Abelian group, its irreducible representations are one dimensional: in fact let $e^{i\alpha} \to U(\alpha)$ be an irreducible representation over a linear space L; then for any β

$$U(\alpha)\, U(\beta) = U(\beta)\, U(\alpha)$$

The corresponding conserved current is the charge-density current, and point a) expresses charge conservation. However being this group Abelian, its irreducible representations are 1-dimensional, so that each resulting multiplet contains only one particle. In this case there are no consequences other than mere charge conservation.

Conservation of any additive charge-like quantity (e.g. baryonic number or hypercharge) can be obtained in this way.

7.4. On the basis of previous considerations it should be clear how things actually go. Strong interactions display additive conservation laws, and moreover the great variety of strongly interacting particles (hadrons) seems to divide naturally into sets of particles with very analoguos properties (for example π^+, π^-, π^0; K^+, K^0; p, n and so on). One takes these experimental facts as an indication that the underlying dynamics possesses a non abelian symmetry group. Then one tries with some Lie group and compares with experiments the relations which can be found in the way outlined before.

Imagine a world in which only proton, neutron, π^+, π^-, π^0, are present, as was the situation at the time when the isotopic spin was introduced.

The mass spectrum of these particles provides a very clear evidence for the existence of a non Abelian symmetry group, whose representation on one particle states splits up into two components, the pion and the nucleon. Inside each multiplet, particles are distinguished by only one quantum number (charge), so that we are led to a non Abelian Lie group of rank one. As we will see later there is only one compact simple group of this kind, i.e. SU_2. Its Lie algebra is spanned by three elements T_1, T_2, T_3, satisfying the product rules of angular momentum, and its irreducible representations are labeled by a number $T \geqq 0$ such that:

i) $2T$ is an integer:

ii) the dimension of the representation is $2T + 1$;

iii) the spectrum of T_3 consists of numbers $T, T - 1, \cdots, - T$.

We have then to assign the nucleon to the $T = 1/2$, and the pion to the $T = 1$ representations. T is called isotopic spin (I-spin) [*15*].

If we assume, as a convention, that particles correspond to eigenstates of T_3, then T_3 has to be connected to the charge operator. In fact the following relation holds for pions and nucleons:

$$Q - \frac{1}{2} N = T_3 \qquad (N = \text{baryonic number}). \tag{5}$$

In this context we have two additive conservation laws: charge and baryonic number. The latter is derived as invariance under a gauge group, whereas charge conservation is included in isotopic spin conservation. Note that T_1 and T_2 do not commute neither with T_3 nor with charge, so that they do not correspond to observable quantities.

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so that (Schur's lemma) $U(\alpha) = \lambda(\alpha) \cdot 1$. Hence L has to be one-dimensional. Irreducible representations are of the form

$$e^{i\alpha} \to \lambda_k(\alpha) = e^{ik\alpha}, \quad \text{where } k \text{ is an integer.}$$

Consider now a $\pi - N$ system. Under SU_2 it transforms as an element of the tensor product of the I-spin 1 and I-spin $^1/_2$ representations. It is well known that this product splits up into an I-spin $^3/_2$ and an I-spin $^1/_2$ representation. If we want to study a scattering process of the type

$$\pi + N \to \pi' + N' \tag{6}$$

we have to evaluate matrix elements of the type:

$$\langle \pi N \mid S \mid \pi' N' \rangle \tag{7}$$

where S is invariant under SU_2. Writing

$$\mid \pi N \rangle = C^3_{\pi N} \mid {}^3/_2\, T_3 \rangle + C^1_{\pi N} \mid {}^1/_2\, T_3 \rangle$$

with the aid of Clebsch-Gordan coefficients, (7) writes as:

$$C^3_{\pi N}\, C^3_{\pi' N'} \langle {}^3/_2\, T_3 \mid S \mid {}^3/_2\, T'_3 \rangle + C^3_{\pi N}\, C^1_{\pi' N'} \langle {}^3/_2\, T_3 \mid S \mid {}^1/_2\, T'_3 \rangle + \\ + C^1_{\pi N}\, C^1_{\pi' N'} \langle {}^1/_2\, T_3 \mid S \mid {}^1/_2\, T'_3 \rangle + C^1_{N \pi}\, C^3_{\pi' N'} \langle {}^1/_2\, T_3 \mid S \mid {}^1/_2\, T'_3 \rangle . \tag{8}$$

Using the analysis of sect. 6.2, we see that

$$\langle {}^3/_2\, T_3 \mid S \mid {}^3/_2\, T'_3 \rangle = \delta_{T_3\, T'_3} A^3$$
$$\langle {}^1/_2\, T_3 \mid S \mid {}^1/_2\, T'_3 \rangle = \delta_{T_3\, T'_3} A^1 ,$$

where A^3 and A^1 depend only upon space-time labels. All the other matrix elements vanish. In conclusion we can express the amplitudes of all processes like (6) in terms of only two amplitudes which are function of space-time variables, but do not depend anymore on the charge variables.
It is a well known fact that experimentally at an energy near 190 MeV for the incident pion the amplitude A^3 greatly dominates: neglecting A^1 we find at that energy a well determined ratio for the following processes:

a) $\pi^+ + p \to \pi^+ + p$
b) $\pi^- + p \to \pi^- + p$
c) $\pi^- + p \to \pi^0 + n$.

Rate a: Rate b: Rate c = 9:1:2 which is well verified experimentally.
Since isotopic spin has been introduced many other hadrons have been found, together with another additive conservation law: hypercharge (Y) conservation. However all hadrons still fit well into isomultiplets when relation (5) is modified as

$$Q - \frac{1}{2} Y = T_3$$

and all the experimental findings are consistent with the assumption that strong interactions are invariant under SU_2 [*16*]. In this context, as for baryonic number, hypercharge conservation is considered to derive from invariance under a gauge group, quite independently from isotopic spin conservation.

This situation can be summarized stating that the symmetry group for strong interactions is the direct product[13]) of two gauge groups (N and Y-conservation) times SU_2 (I-spin conservation). As a consequence N, (T, T_3) and Y quantum numbers appear in a completely uncorrelated way.

7.5. Unitary symmetry models

In unitary symmetry models one tries to derive T_3 and Y conservation from invariance under a group which does not break into the direct product $SU_2 \times U_1(Y)$, introducing in this way relations between particles belonging to isomultiplets with different hypercharge. It is a fact that as yet nobody has succeeded in extending such procedure to include N-conservation which, as before, is derived from a separated gauge group [*17*].

The feeling for such a higher symmetry is not strongly substantiated at first sight by experimental evidence. In fact according to results derived in sect. 7.3) particles would be organized into supermultiplets, (i.e. irreducible representations) behaving as elementary objects under strong interactions; but now inside same supermultiplets there would appear particles differing by Y as well as by T_3. This is in conflict with the experimental evidence in that $\Delta m/m$ between particles differing by Y are quite large and not imputable to non strong interactions (for example $m_\Lambda - m_N \cong 175$ MeV, $m_\Xi - m_N \cong 380$ MeV). Hence we must conclude that the idea of a higher symmetry in the sense above specified, cannot be literally applied.

Nevertheless the following interpretation has been proposed: there is a symmetrical component in strong interactions which is responsible of the gross structure of particles world; in addition there is a weaker component to be treated as a perturbation responsible of the departures from the exact symmetry. It is understood that both components are charge independent as well as strangeness conserving.

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[13]) Given two groups G_1 and G_2, their direct product $G_1 \times G_2$ is defined as the set of ordered pairs $(g_1, g_2)(g_i \in G_i)$ with the multiplication law

$$(g_1, g_2)(f_1, f_2) = (g_1 f_1, g_2 f_2).$$

This definition satisfies all the required axioms. Given a representation $g_i \to U(g_i)$ of G_i on the linear spaces L_i, we can find a representation of $G_1 \times G_2$ in the direct product $L_1 \times L_2$ as follows

$$(g_1, g_2) \to U(g_1) \otimes U(g_2) = U(g_1, g_2)$$

and it can be shown that all the representations of $G = G_1 \times G_2$ can be put into this form. $U(g_1, g_2)$ is irreducible if and only if $U(g_1)$ and $U(g_2)$ are. In our particular case we have the group

$$SU_2 \times U_1(N) \times U_1(Y)$$

specified by the triplets (α, β, g) (α, β real, $g \in SU_2$).
In a space spanned by $2T + 1$ vectors

$$|N, Y; T, T_3), \ -T \leq T_3 \leq T, N, Y \text{ fixed}$$

an irreducible representation of $SU_2 \times U_1(N) \times U_1(Y)$ has the form:

$$(\alpha, \beta, g) \to e^{i\beta N} e^{i\alpha Y} D^{(T)}(g)$$

where $D^{(T)}(g)$ are matrices defining an irreducible representation of SU_2.

It should be noted that the weaker component has not been till now satisfactory identified. However the idea of a simmetry breaking interaction treated as first order perturbation, has provided us with corrections to the predictions derived from a pure symmetrical model, which are consistent with experimental findings.

7.6. Previous reasoning obviously do not indicate us what the symmetry group for strong interactions actually is. Following general requirements however seem to be quite reasonable, and are usually imposed on possible candidates:

i) this group must be a Lie group. In fact we want to identify additive conserved quantities such as T_3 and Y with its infinitesimal generators;

ii) is must be compact: this assures that its irreducible unitary representations are finite-dimensional, so that wen can fill up resulting supermultiplets with a finite number of particles (see sect. 2.2);

iii) it must be semi-simple (see sect. 4.4). This restriction is mainly due to practical reasons: for semisimple Lie groups in fact there is a complete mathematical theory, which is not the case non for semisimple groups;

iv) the rank of the group, i.e. the rank of its Lie algebra, must be two, because we require two conserved commuting quantities i.e. T_3 and Y;

v) is must contain a subgroup isomorphic to SU_2 in order to recover the isotopic spin symmetry. Actually this does not bear any restriction in that any semisimple Lie group has this property (see later sect. 9.7b).

To construct a concrete theory we need at this point a characterization of Lie groups as well as a classification of their irreducible representations.

In next sections we will study these topics with some detail.

8. Structure of Semisimple Lie Algebras

In sect. 4 we studied the relations between Lie groups and Lie algebras, and the conclusion achieved was that there is a one-to-one correspondence between Lie algebras and Lie groups, so that instead of studying Lie groups one can study the corresponding Lie algebras and their representations, which is more convenient by a mathematical point of view.

8.1. We have given in sect. 4.4 the definition of n-dimensional Lie algebra of rank r. At the same time we noted that the mapping

$$x \rightarrow \mathrm{ad}\, x \qquad x \in \mathscr{L}$$

$$\mathrm{ad}\, x(y) = [x, y]$$

is a representation of $\mathscr{L}$ called its regular representation.

In terms of it we can define in $\mathscr{L}$ a bilinear form as follows

$$(x, y) = \mathrm{Tr}\,(\mathrm{ad}x, \mathrm{ad}y)\,. \tag{1}$$

Obvious properties are (α = real number)

$$(\alpha x, y) = \alpha (x, y)\,; \qquad (x + y, z) = (x, z) = (y, z) \qquad (1\ )$$

$$(x, y) = (y, x)$$

$$([z, x], y) = -(x, [z, y])\,.$$

The following very important theorem has been proved by Cartan:
$\mathscr{L}$ is semisimple if and only if (x, y) is not degenerate, i.e. if $(x, y) = 0$ for any $y \in \mathscr{L}$, implies $x = 0$.
This criterion is essential in the classical theory of semisimple Lie algebras. Furthermore the most important results of this theory heavily rest on the possibility of diagonalizing operators $\operatorname{ad} x$ where x runs over the elements of a Cartan subalgebra $\mathscr{C}$ of $\mathscr{L}$. Now in $\mathscr{L}$ there are surely eigenvectors of $\operatorname{ad}(x)$ (in fact for each element $y \in \mathscr{C}$ we have:

$$\operatorname{ad} x(y) = [x, y] = 0 \qquad x, y \in \mathscr{C}$$

so that $\mathscr{C}$ is an eigenspace of $\operatorname{ad}(x)$ belonging to the eigenvalue zero) but in general in $\mathscr{L}$ (which is a real vector space) a complete system of such eigenvectors does not exists (see later the example reported in sect. 8.7a). The way out of this difficulty is to englarge $\mathscr{L}$ to a complex Lie algebra in which structure theory can be easily carried out. From it, as we shall see later, corresponding results for the real semisimple Lie algebras associated with compact Lie groups can be deduced.

8.2. Complexification

If $\mathscr{L}$ is a real semisimple Lie algebra, we can construct its complex extension $\mathscr{L}_c$ by choosing a basis x_k in $\mathscr{L}$ and considering the set of all linear combinations of this basis, with the product between two elements defined as follows. If

$$x = \sum_{k}^{1,n} \lambda_k x_k \quad (\lambda_k, \mu_h = \text{complex numbers})$$

$$y = \sum_{k}^{1,n} \mu_h x_h$$

then

$$[x, y] = \sum_{h,k}^{1,n} \lambda_k \mu_h [x_k, x_h] = \sum_{hks}^{1,n} \lambda_h \mu_h C_{kh}^s x_s$$

where C_{kh}^s are the structure constants relative to the basis choosen in $\mathscr{L}$. This product satisfies condition 4.4ii), and obviously this definition of $\mathscr{L}_c$ does not depend upon the particular basis choosen. In $\mathscr{L}_c$ the form (x, y) is simply

$$(x, y) = \sum_{h,k} \lambda_h \mu_k (x_h, x_k).$$

This form is not degenerate if and only if $\det(x_h, x_k) \neq 0$; this condition is the same whether we consider $\mathscr{L}$ or $\mathscr{L}_c$, so that if $\mathscr{L}$ is semisimple so is $\mathscr{L}_c$. In addition the rank of $\mathscr{L}_c$ is the same of that of $\mathscr{L}$.

8.3. Structure theory

We will be content here to state without proof all results relevant for applications. A complete derivation can be found in the excellent book by Jacobson [*18*][14]).

[14]) An easier treatment can be found in [*19*]. See in addition the very readable paper by Racah [*20*].

In the following by $\mathcal{L}$ we mean a complex semisimple n-dimensional Lie algebra of rank r and by $\mathscr{C}$ one of its Cartan subalgebras.

I. Any operator $\mathrm{ad}\,h$, $h \in \mathscr{C}$, is diagonalizable in $\mathcal{L}$. If $h_1, \ldots h_r$ is a basis in $\mathcal{L}$, and $e_{\alpha_1 \ldots \alpha_r}$ is a common eigenvector of ad h_i's(ad $h_i(e_{\alpha_1 \ldots \alpha_r}) = \alpha_i e_{\alpha_1 \ldots \alpha_r}$), we have for any element $h \in \mathscr{C}$:

$$h = \sum_i^{i,r} \lambda_i h_i$$

$$\mathrm{ad}\,h\, e_{\alpha_1 \ldots \alpha_r} = \left(\sum_i \lambda_i \alpha_i\right) e_{\alpha_1 \ldots \alpha_r}$$

so that it sufficies to sonsider only the diagonalization of operators $\mathrm{ad}\,h_i$.

II. h_i's can be choosen in such a way that the α_i are all real. Upon introducing the notation $\alpha = (\alpha_1 \ldots, \alpha_r)$ we write

$$\mathrm{ad}\,h_i e_\alpha = \alpha_i e_\alpha .$$

The real r-components object α is called a root vector, or simply a root. $\mathcal{L}$ splits up into a direct sum of common eigenspaces of $\mathrm{ad}\,h_i$ $(i = 1, 2 \ldots r)$

$$\mathcal{L} = \mathcal{L}_0 \bigoplus_\alpha \mathcal{L}_\alpha \tag{2}$$

where the direct sum runs over all non vanishing roots and $\mathcal{L}_0$ is the eigenspace belonging to the root $(0, \ldots, 0)$. Obviously $\mathcal{L}_0$ contains $\mathscr{C}$, but a stronger result holds, namely

III.
$$\mathcal{L}_0 = \mathscr{C}, \tag{3}$$

and furthermore each $\mathcal{L}_\alpha (\alpha \neq 0)$ is one dimensional.

It follows the there are $n - r$ non zero roots.

We can extend the basis $\{h_i\}$ to a basis in $\mathcal{L}$, adding to these elements all the vectors e_α where α is a non zero root and e_α is a vector spanning $\mathcal{L}_\alpha$.

IV. Consider the restriction of the trace form (1) to $\mathscr{C}$. It is determined by the matrix:

$$g_{ij} = (h_i, h_j) = \mathrm{Tr}(\mathrm{ad}\,h_i\, \mathrm{ad}\,h_j), \qquad i, j = 1, 2, \ldots r. \tag{4}$$

In the basis $\{h_i, e_\alpha\}$, the operators

$$\mathrm{ad}\,h_i\, \mathrm{ad}\,h_j$$

are diagonal, and have eigenvalues equal to $\alpha_i \alpha_j$. Hence

$$g_{ij} = \sum_\alpha \alpha_i \alpha_j \quad \text{(the sum runs over all roots)} \tag{5}$$

which implies g_{ij} to be a real symmetric matrix.

As a consequence g_{ij} can be diagonalized with a real orthogonal substitution:

$$g'_{ij} = \sum_{kh} A_{ik} A_{jh} g_{kh} = \lambda_j \delta_{ij} .$$

Upon introducing

$$h'_i = \sum_k A_{ik} h_k$$

which corresponds to a change of basis in $\mathscr{C}$, we see that:

$$\mathrm{ad}\, h'_i e_\alpha = \left(\sum_k A_{ik}\alpha_k\right) e_\alpha = \alpha'_i e_\alpha$$

and α'_i also are real numbers. Then:

$$\mathrm{Tr}(\mathrm{ad}\, h'_i\, \mathrm{ad}\, h'_j) = \sum_\alpha \alpha_i \alpha_j = \sum_{h,k} A_{ik} A_{jh} g_{hk} = g'_{ij} = \lambda_i \delta_{ij}.$$

Hence

$$\lambda_i = g_{ii} = \sum_\alpha (\alpha'_i)^2 \geqq 0.$$

Suppose that for some value of i, $\lambda_i = 0$. Then $\alpha'_i = 0$ for any α, so that

$$[h'_i, e_\alpha] = 0.$$

Moreover

$$[h'_i, h'_j] = 0,$$

so that $\mathrm{ad}\, h'_i$ is represented in the basis $\{h'_i, e_\alpha\}$ by the null matrix.
This has the consequence

$$(h'_i, x) = 0 \quad \text{for any} \quad x \in \mathscr{L}$$

i.e. (by Cartan's criterion) $h'_i = 0$ which is excluded.
In conclusion we see that g_{ij} is a non singular, positive definite, real matrix. We will indicate with g^{ij} its inverse

$$\sum_k g_{ik} g^{kj} = \delta_{ij}.$$

With the aid of this metric tensor we define a scalar product between roots:

$$(\alpha, \beta) = \sum_i \alpha^i \beta_i = \sum_i \alpha_i \beta^i = \sum_{ij} \alpha_i \beta_j g^{ij}. \qquad (6)$$

The difference between $\alpha^i = \Sigma\, g^{ij}\alpha_j$ and α_i can be removed by performing a real linear transformation on h_i's, which reduces g_{ij} into the form δ_{ij}. In this basis the scalar product between roots is written as

$$(\alpha, \beta) = \sum_i \alpha_i \beta_i.$$

In the basis $\{h_i, e_\alpha\}$ part of the multiplication rules are defined in terms of roots

$$[h_i, h_j] = 0$$

$$[h_i, e_\alpha] = \alpha_i e_\alpha.$$

We will see that the same applies to all multiplication rules between h_i and e_α, in the sense that roots determine all the structure constant relative to the basis $\{h_i, e_\alpha\}$. This depends on peculiar properties of roots, which are the very heart of structure theory.

Properties of roots.

In the following $k\alpha$ and $\alpha + \beta$ are defined as

$$k\alpha = (k\alpha_1, \ldots, k\alpha_r)$$

$$\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_r + \beta_r).$$

V. If α is a non zero root, then $k\alpha$ is a root (k = real number) if and only if $k = \pm 1{,}0$.

Hence the $n - r$ non zero roots are distributed in pairs α, $-\alpha$. (From this we see that $n - r$ is an even integer for any semisimple Lie algebra).

VI. Any two non zero roots $\alpha, \beta, (\alpha + \beta \neq 0)$ uniquely determine two non negative integers r, q such that

$$\beta - r\alpha, \beta - (r - 1)\alpha, \ldots, \beta, \beta + \alpha, \ldots \beta + q\alpha$$

are the only non zero roots of the form $\beta + k\alpha$. This serie of roots is called the α-string containing β. Interchanging α and β we obtain two other integers q', r' characterizing the β-string containing α.

Numbers r, q satisfy the condition:

$$r - q = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} \qquad (r' - q' = 2\frac{(\alpha, \beta)}{(\beta, \beta)}. \tag{7}$$

Being in addition

$$-r \leqq q - r \leqq q \quad (-r' \leqq q' - r' \leqq q')$$

we obtain that if α and β are non zero roots, then

$$\beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha, \qquad \alpha - \frac{2(\alpha, \beta)}{(\beta, \beta)}\beta \tag{8}$$

are non zero roots. The first (second) os obtained by reflecting $\beta(\alpha)$ with respect to the plane orthogonal to $\alpha(\beta)$.

The general feature of the multiplication table can be understood in terms of the following relation:

$$\mathrm{ad}h_i[e_\alpha, e_\beta] = [h_i, [e_\alpha, e_\beta]] = (\alpha_i + \beta_i)[e_\alpha, e_\beta] \tag{9}$$

which is a simple consequence of Jacobi identity.

We distinguish three cases:

a) $\alpha + \beta \neq 0$, $\alpha + \beta$ is not a root.

In this case eq. (9) implies, $[e_\alpha, e_\beta] = 0$. Otherwise $[e_\alpha, e_\beta]$ would be an eigenvector of h_i and $\alpha + \beta$ would be a root.

b) $\alpha + \beta = 0$.

Hence ad $h_i[e_\alpha, e_{-\alpha}] = 0$ so that $[e_\alpha, e_\beta] \in \mathscr{C}$ and we can write:

$$[e_\alpha, e_{-\alpha}] = \sum_i \lambda^i h_i.$$

VII. It is possible to normalize $e_\alpha, e_{-\alpha}$ so that $(e_\alpha, e_{-\alpha}) = \mathrm{Tr}(\mathrm{ad}\,e_\alpha\, \mathrm{ad}\,e_{-\alpha}) = 1$. With this choice it results:

$$\lambda^i = \alpha^i = \sum_j g^{ij} \alpha_j$$

$$[e_\alpha, e_{-\alpha}] = \sum_j g^{ij} \alpha_j h_i = \sum_i \alpha^i h_i. \tag{10}$$

We note that this normalization determines e_α and $e_{-\alpha}$ up to factors $d_\alpha, d_{-\alpha}$ such that $d_\alpha d_{-\alpha} = 1$.
c) $\alpha + \beta \neq 0$, $\alpha + \beta$ is a root. By eq. (9) and by unidimensionality of $\mathcal{L}_{\alpha+\beta}$ it follows:

$$[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}.$$

One can show that

$$N_{\alpha,\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha} = -N_{\beta\alpha} \tag{11}$$

and in addition, by disposing of factors $d_\alpha, d_{-\alpha}$ in $e_\alpha, e_{-\alpha}$, one can assume

$$N_{\alpha\beta} = -N_{-\alpha,-\beta}. \tag{12}$$

In this case

VIII.
$$N^2_{\alpha\beta} = \frac{q(r+1)}{2} (\beta, \beta) \tag{13}$$

where q, and r are the integers determining the α-string containing β. We see that when $\alpha + \beta$ is a root, $N_{\alpha\beta} \neq 0$.
This relation determines $N_{\alpha\beta}$ up to a sign which must be chosen so to satisfy (11), (12).
Collecting all these results, we can write the complete multiplication table relative to the basis $\{h_i, e_\alpha, e_{-\alpha}\}$:

$$[h_i, h_j] = 0$$

$$[h_i, e_{\pm\alpha}] = \pm \alpha_i e_{\pm\alpha}$$

$$[e_\alpha, e_{-\alpha}] = \sum_j \alpha^i h_i = \sum_{ij} \alpha_i h_i g^{ij} \tag{14}$$

$$[e_\alpha, e_\beta] = \begin{cases} (\alpha \neq -\beta,\ \alpha + \beta \quad \text{is not a root}) = 0 \\ (\alpha \neq -\beta,\ \alpha + \beta \quad \text{is a root}) = N_{\alpha\beta}\, e_{\alpha+\beta}. \end{cases}$$

All structure constants relative to this basis are determined by roots and are all real. In addition a linear non singular transformation on the h_i's does not change the form of these products. In fact if we have

$$h_i' = \sum_j A_{ij} h_j, \quad h_j = \sum_i = (A^{-1})_{ji} h_i',$$

then

$$[h_i', e_{\pm\alpha}] = \sum_j A_{ij} [h_j, e_{\pm\alpha}] = \pm \Big(\sum_i A_{ij} \alpha_j\Big) e_{\pm\alpha} = \alpha_i' e_{\pm\alpha}$$

$$[e_\alpha, e_{-\alpha}] = \sum_i \alpha^i h_i = \sum_{i,l} \alpha^i (A^{-1})_{il} h_l' = \sum_l \alpha'^l h_l',$$

where

$$\alpha'_i = \Sigma_j A_{ij}\alpha_j\,; \quad \alpha'^l = \sum_i \alpha\,(A^{-1})_{il}\,,$$

so that α_i and α^j transform respectively as the covariant and contravariants components of a vector. g_{ij} transforms as a covariant tensor:

$$g'_{ij} = (h'_i, h'_j) = \sum_{l,k} A_{il} A_{ik} g_{lk}\,, \tag{15}$$

so that the scalar product between roots is dependent upon the particular basis choosen in $\mathscr{C}$. From this it follows that although there are in $\mathscr{C}$ bases in which α'_is are complex numbers, in any case (α, α) is a positive number.

By (14) we see that the roots of a Lie algebra, determine uniquely its structure. Hence a classification of semisimple Lie algebras of rank r is equivalente to find all sets of r-dimensional real vectors which satisfy V, VI, (7). This is the argument of next section.

8.4. To begin with we introduce now an ordering between roots in the following way.

Given two roots α and β, α is said to be greater than β if the first non vanishing component of $\alpha - \beta \equiv (\alpha_i - \beta_i)$ is a positive number. In particular a root is positive if it is greater than zero.

Of course this ordering depends on the basis choosen in $\mathscr{C}$ and it is the same that the ordering of words in a dictionary.

We introduce another useful concept. A root α is simple if:

a) α is a positive root,

b) α cannot be written as sum of two positive roots.

Two important properties of simple roots are nearly immediate.

If α and β are simple roots, then:

i) $\alpha - \beta$ is not a root. If $\alpha - \beta$ were a positive root, then α would be equal to $(\alpha - \beta) + \beta$, i.e. would not be simple. Conversely if $\alpha - \beta$ were a negative root, then $\beta - \alpha$ would be positive, and β would be equal to $(\beta - \alpha) + \alpha$, i.e. β would not be simple;

ii) $(\alpha, \beta) \leqq 0$. By i) if we consider the α-string containing β, we see that $r = 0$, so that

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = r - q = -q \leqq 0\,.$$

The usefulness of considering simple roots lies in the following theorem:

IX. There are exactly r linearly independent simple roots which we will indicate with $\alpha^{(1)}, \ldots \alpha^{(r)}$. Furthermore any positive root can be written as a linear combination of simple roots with non negative integers as coefficients.

X. if $\alpha > 0$ is a non simple root, there exists a simple root $\alpha^{(k)}$ such that $\alpha - \alpha^{(k)}$ is a positive root.

We will see later that properties IX), X) enable one to construct all roots starting form simple roots. This limits further analysis only to simple roots.

From (7) we can derive very severe restriction on the angle between two roots as well as on the ratio of their lenghts. In fact it is:

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = m\,, \quad \frac{2(\alpha, \beta)}{(\beta, \beta)} = n$$

i.e.

$$\frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = \frac{m \cdot n}{4} = \cos^2 \varphi_{\alpha\beta} \leqq 1;$$

if $m, n \neq 0$

$$\frac{(\alpha, \alpha)}{(\beta, \beta)} = \frac{n}{m}.$$

When α and β are simple roots, m and n are non positive by 8.4 ii) and furthermore $\alpha \neq \beta$ so that the only possibilities we are left with are:

m	n	$\varphi_{\alpha\beta}$	$(\alpha, \alpha)/(\beta, \beta)$
-1	-1	120°	1
-1	-2	135°	2
-1	-3	150°	3
-2	-1	135°	$1/2$
-3	-1	150°	$1/3$
0	0	90°	arbitrary

In addition in the latter case ($\varphi_{\alpha\beta} = 90°$) neither $\alpha - \beta$, nor $\alpha + \beta$ are roots (see properties 8.4i, ii).
8.5. On the basis of stated properties of simple roots, we illustrate the classification of (complex) semisimple Lie algebras.
Let us first examine the case in which simple roots split up into groups of simple roots, such that any root of each group is orthogonal to all roots belonging to different groups, whereas inside each group there is no root orthogonal to all the other ones:

$$\alpha^{(1)}, \ldots \alpha^{(r_1)}; \quad \beta^{(1)}, \ldots \beta^{(r_2)}; \quad \gamma^{(1)}, \ldots \gamma^{(r_m)}$$

$$r_1 + r_2 + \cdots + r_m = r.$$

The whole root diagram splits up into mutually orthogonal parts $\alpha, \ldots, \beta \ldots; \ldots;$ $\gamma, \ldots$ and in addition $\alpha \pm \beta \ldots; \ldots \alpha \pm \gamma, \ldots; \beta \pm \gamma \ldots$ are not roots, so that for the corresponding $e_\alpha, \ldots; e_\beta, \ldots; e_\gamma \ldots$ we have

$$[e_\alpha, e_\beta] = 0, \quad [e_\alpha, e_\gamma] = 0 \quad \text{and so on.}$$

Furthermore one can choose in $\mathscr{C}$ a basis of h_i's which also decomposes into groups of vectors $\{h_1^{(1)}, \ldots h_{r_1}^{(1)}; h_2^{(2)}, \ldots h_{r_2}^{(2)}; \ldots$ such that for example

$$[h_i^{(1)}, e_\beta] = 0 \ldots$$

$$[h_i^{(1)}, e_\gamma] = 0 \ldots$$

Summarizing we see that the basis $\{h_i e_\alpha, e_{-\alpha}\}$ can be decomposed into the direct sum of bases $\{h_i^{(1)}, e_\alpha, e_{-\alpha}\}$, $\{h_j^{(2)}, e_\beta, e_{-\beta}\}, \ldots \{h_l^m, e_\gamma, e_{-\gamma}\}$ such that all products between elements of different groups vanish.
Each linear manifold spanned by such bases is evidently a subalgebra in $\mathscr{L}$ and it is even an ideal, so that the existence of roots orthogonal to all the others implies $\mathscr{L}$ to be not simple. The converse is also true, i.e.

XI. Necessary as well as sufficient condition for a semisimple Lie algebra to be simple is that $\mathcal{L}$ has no simple root orthogonal to all the others.
From XI and from the previous considerations it follows that any semisimple Lie algebra is the direct sum of simple Lie algebras (Weyl's theorem) so that the classi fication of semisimple Lie algebras is reduced to that of simple one.

Classification of simple Lie algebras

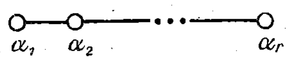
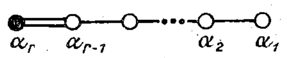
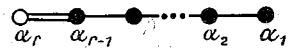
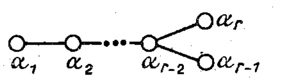
As it should be clear from last section, the problem of determining all simple Lie algebras of a fixed rank r, is equivalent to finding all sets of r simple roots satisfying V, VI, (7), and the condition, that no one of them is orthogonal to all the others.
The essential results of the structure theory can be formulated as following.
XII. Lenght of simple roots can assume at most two values.
Keeping this in mind, the set of simple roots of a simple Lie algebra can be conveniently described in a graphical way introduced by E. B. DYNKIN:

(1) to any simple root we associate a circle:

(2) two circles are connected by one, two, or three lines when the angle between corresponding roots is respectively 120°, 135°, or 150°.
If the roots are orthogonal, circles are not connected.

(3) Circles corresponding to shorter roots are blackened.

The only simple algebras are then defined by following diagrams (for any fixed r):

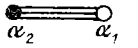
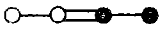
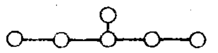
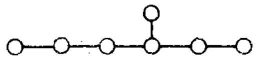
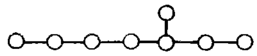
Nome of the algebra	Dynkin diagram	Dimensionality	Remarks
A_r		$r(r+2)$	$r \geqq 1$
B_r		$r(2r+1)$	$r \geqq 2$
C_r		$r(2r+1)$	$r \geqq 2$
D_r		$r(2r-1)$	$r \geqq 3$

These algebras are all distinct when $r \geqq 4$, wherease we note that:

i) when $r = 1$ there is only one simple Lie algebra, i.e. A_1;

ii) when $r = 2$, Dynkin diagrams of B_2 and C_2 are identical, i.e. B_2 and C_2, having same dimensionality and same structure constants are identical;

iii) when $r = 3$, A_3 and D_3 have the same Dynkin diagram so that again $A_3 = D_3$.
Apart from these four classes, there are five exceptional Lie algebras, named G_2, F_4, E_6, E_7, E_8, defined by the following diagrams:

Name	Diagram	Dimensionality
G_2		14
F_4		52
E_6		78
E_7		133
E_8		248

In particular there are only three distinct Lie algebras of rank two, i.e. A_2, $C_2 = B_2$, G_2.

8.6. Classification of simple, compact Lie groups

As said in sect. 7.5 these groups are of the main concern in unitary symmetry models. To any of them we can uniquely associate a real Lie algebra $\mathscr{L}_r$, and it is remarkable that, as showed by H. WEYL, the compactness of the group reflects in $\mathscr{L}_r$ in that its trace form (1) is negative definite. In view of this circumstance, $\mathscr{L}_r$ itself is called a compact (real) Lie algebra.

To carry out structure theory it has been convenient to consider complex Lie algebras. Now, whereas any real algebra $\mathscr{L}_r$ uniquely define its complex extension $\mathscr{L}_c$, the converse is not true, in that the same $\mathscr{L}_c$ can be obtained starting from different real Lie algebras i.e. from different Lie groups (this is for example the case of R_3 and of the 3-dimensional Lorentz group, which have the same complex Lie algebra A_1).

However, as again has been showed by H. WEYL, for any semisimple complex Lie algebra $\mathscr{L}_c$, there is essentially one real semisimple compact Lie algebra whose complex extension is $\mathscr{L}_c$. This has the meaning that in $\mathscr{L}_c$ there exists a basis such that:

i) all its structure constants are real;

ii) the real Lie algebra spanned by this basis is compact.

In particular starting from the canonical basis $\{h_i, e_\alpha, e_{-\alpha}\}$ it can be easily shown that the basis

$$f_i = -ih_i; \quad f_\alpha = -i(e_\alpha + e_{-\alpha}); \quad g_\alpha = -(e_\alpha - e_{-\alpha}) \tag{16}$$

(α runs over positive roots)

is compact[15]).

8.7. Examples

a) SU_2

We have seen in sect. 4.1 that the real Lie algebra associated to $SU_2(R_3)$ is spanned by three elements I_1, I_2, I_3 with the product rules:

$$[I_1, I_2] = I_3, \quad [I_2, I_3] = I_1, \quad [I_3, I_1] = I_2.$$

[15]) To verify that $(x, x) < 0 (x \neq 0)$ whenever x is a real linear combination of the elements (16), one has simply to use the orthogonality relations $(h_i, e_\alpha) = 0$, $(e_\alpha, e_\beta) = \delta_{-\alpha,\beta}$ which follow from (1') and from the normalization condition $(e_\alpha, e_{-\alpha}) = 1$.

This is a 3-dimensional simple Lie algebra of rank 1. For the general element $x = \sum_i c_i I_i (c_i \text{ real})$ it is easy to see that $\mathrm{Tr}(\mathrm{ad}x\,\mathrm{ad}x) < 0$ as it must be, being SU_2 compact. Let us choose I_3 as the element spanning $\mathscr{C}$. Finding roots is equivalent to solve the following eigenvalue equation (with $\alpha \neq 0$):

$$\mathrm{ad}\,I_3(x_\alpha) = [I_3, x_\alpha] = \alpha x_\alpha, \qquad x_\alpha = c_1 I_1 + c_2 I_2 + c_3 I_3,$$

which is equivalent to:

$$\begin{cases} c_1 = \alpha c_2, \\ c_2 = -\alpha c_1 \end{cases} \quad \text{i.e.} \quad \alpha = \pm i, \quad c_2 = \mp i c_1 .$$

If we introduce the elements $I_\pm = I_1 \pm i I_2$ (which are in the complexification of our algebra!), we have

$$[I_3, I_\pm] = \mp i I_\pm, \quad [I_+, I_-] = -2 i I_3$$

so that, by posing

$$h_3 = i I_3; \quad e'_\pm = \frac{i}{2} I_\pm$$

we obtain the product rule

$$[h_3, e'_\pm] = \pm e'_\pm .$$

We observe that $\mathrm{Tr}(\mathrm{ad}h_3\,\mathrm{ad}h_3) = 2$ so that the metric tensor (which reduces to a number g) is $g = 2$. The contravariant component of the single positive root is

$$\alpha^1 = \frac{1}{2}\alpha_1 = \frac{1}{2}$$

so that we have

$$[e'_+, e'_-] = \frac{1}{2}[e'_+, e'_-] \quad h_3 = \frac{1}{2} h_3$$

being $(e'_+, e'_-) = 1$ as one can easily verify.
Root space is one-dimensional, and we have two non zero roots: ± 1, and one simple root. The corresponding Dynkin diagram is made of a single circle, so that this Lie algebra is just A_1.
Usually as basis are taken the elements

$$h_3, e_\pm = \sqrt{2}\, e'_\pm \quad ((e_+, e_-) = 2)$$

whose product rules are

$$[h_3, e_\pm] = \pm e_\pm$$
$$[e_+, e_-] = h_3$$

and this we will do in the following.
b) Recalling the product rules (14) which are the canonical rules for any semisimple Lie algebra $\mathscr{L}$, we see that the elements of $\mathscr{L}$, defined as:

$$h'_\alpha = \frac{\Sigma_i \alpha^i h_i}{(\alpha, \alpha)} \quad e'_{\pm\alpha} = \frac{e_{\pm\alpha}}{(\alpha, \alpha)^{1/2}}$$

satisfy the relations:

$$[h'_\alpha, e'_{\pm\alpha}] = \pm e'_{\pm\alpha}$$

$$(e'_\alpha, e'_{-\alpha}) = h'_\alpha .$$

Hence h'_α, $e'_{\pm\alpha}$ span a subalgebra in $\mathscr{L}$ which is identical with A_1, so that by the considerations made in sect. 4.4 we see that the compact group associated to any $\mathscr{L}$ contains subgroups isomorphic to SU_2.

c) A_2

We start from the Dynkin diagram: O—O
From this we see that there are two simple roots $\alpha^{(1)}$ and $\alpha^{(2)}$ of equal lenghts, making, an angle of 120 degrees:

$$\frac{2(\alpha^{(1)}, \alpha^{(2)})}{(\alpha^{(1)}, \alpha^{(1)})} = \frac{2(\alpha^{(1)}, \alpha^{(2)})}{(\alpha^{(2)}, \alpha^{(2)})} = -1 .$$

Hence the $\alpha^{(1)}$ string containing $\alpha^{(2)}$ consists of the two elements $\alpha^{(2)}$, $\alpha^{(1)} + \alpha^{(2)}$ and the reversed string contains $\alpha^{(1)}$ and $\alpha^{(1)} + \alpha^{(2)}$. A_2 is 8-dimensional and has rank 2, so that we expect six non zero roots at all: in fact they are

$$\pm\alpha^{(1)}, \quad \pm\alpha^{(2)}, \quad \pm(\alpha^{(1)} + \alpha^{(2)}) .$$

The root diagram is a regular hexagon (see fig. 1).

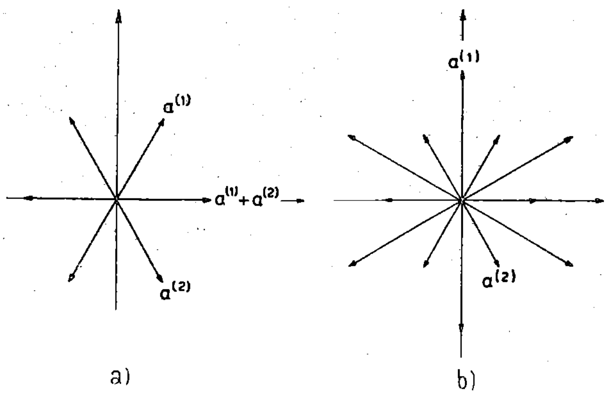


Fig. 1. a) Root diagramm of SU_3 b) Root diagram of G_2

In order to construct explicitly our Lie algebra, we evaluate now covariant components of α's (α_i) in a fixed frame of reference in $\mathscr{C}$. Each choice of the frame will lead us to a well defined set of structure constants in a certain basis $\{h_i, e_\alpha, e_{-\alpha}\}$. The most convenient choice is to refer α's to orthogonal axes, i.e. to axes such that:

$$(h_i, h_j) = g_{ij} = \sum_\alpha \alpha_i \alpha_j = \delta_{ij} .$$

In this case $\alpha^i = \alpha_i$.

In the notation of the fig. 2 (were we have relabeled the roots and for simplicity we use lower indices):

$$\alpha_1 = \left(\frac{1}{\sqrt{3}}, 0\right) \qquad g_{ij} = \sum_\alpha \alpha_i \alpha_j = \delta_{ij}$$

$$\alpha_2 = \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right)$$

$$\alpha_3 = \left(\frac{1}{2\sqrt{3}}, -\frac{1}{2}\right) \qquad (\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = (\alpha_3, \alpha_3) = \frac{1}{3}$$

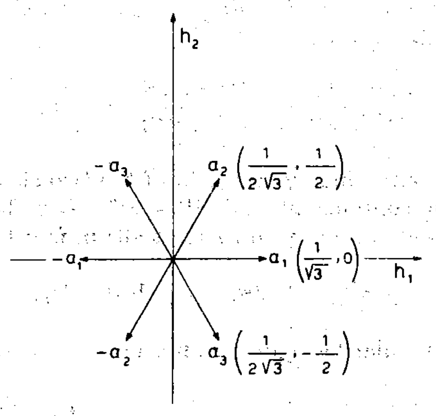


Fig. 2.

In terms of these components, we can write the following product rules:

$$[h_1, e_{\pm 1}] = \pm \frac{1}{\sqrt{3}} e_{\pm 1}, \quad [h_1, e_{\pm 2}] = \pm \frac{1}{2\sqrt{3}} e_{\pm 2}, \quad [h_1, e_{\pm 3}] = \pm \frac{1}{2\sqrt{3}} e_{\pm 3}$$

$$[h_2, e_{\pm 1}] = 0, \quad [h_2, e_{\pm 2}] = \pm \frac{1}{2\sqrt{3}} e_{\pm 2}, \; [h_1, e_{\pm 3}] = \mp \frac{1}{2} e_{\pm 3}$$

$$[e_1, e_{-1}] = \frac{1}{\sqrt{3}} h_1, \quad [e_2, e_{-2}] = \frac{1}{2\sqrt{3}} h_1 + \frac{1}{2} h_2, \quad [e_3, e_{-3}] = \frac{1}{2\sqrt{3}} h_1 - \frac{1}{2} h_2.$$

To complete the multiplication table we need the quantitaties $N_{\alpha\beta}$. By relations (11), (12) we see that we can arbitrarily fix only one sign in $N_{\alpha\beta}$; for example we can fix the sign of N_{23}. Using (13), we have

$$N_{23}^2 = \frac{q(r+1)}{2} (\alpha_3, \alpha_3) = \frac{1}{6}$$

being $q = 1$, $r = 0$. We choose

$$N_{23} = +\frac{1}{\sqrt{6}}$$

so that the remaining multiplication rules are

$$[e_2, e_3] = \frac{1}{\sqrt{6}} e_1 \qquad [e_2, e_{-1}] = -\frac{1}{\sqrt{6}} e_{-3}$$

$$[e_1, e_{-3}] = \frac{1}{\sqrt{6}} e_2 \qquad [e_1, e_{-2}] = -\frac{1}{\sqrt{6}} e_3$$

$$[e_3, e_{-1}] = \frac{1}{\sqrt{6}} e_{-2} \qquad [e_{-3}, e_{-2}] = \frac{1}{\sqrt{6}} e_{-1}.$$

The compact basis is:

$$\lambda_3 = -i\sqrt{3}\,h_1 \qquad \lambda_1 = -i\sqrt{\frac{3}{2}}\,(e_1 + e_{-1}) \qquad \lambda_2 = -\sqrt{\frac{3}{2}}\,(e_1 - e_{-1})$$

$$\lambda_8 = -i\sqrt{3}\,h_2 \qquad \lambda_4 = -i\sqrt{\frac{3}{2}}\,(e_2 + e_{-2}) \qquad \lambda_5 = -\sqrt{\frac{3}{2}}\,(e_2 - e_{-3})$$

$$\lambda_6 = -i\sqrt{\frac{3}{2}}\,(e_3 + e_{-3}) \qquad \lambda_7 = -\sqrt{\frac{3}{2}}\,(e_3 - e_{-3}).$$

We note that this basis differs from that given in (16) only by real factors which do not affect its compactness and have been introduced in order to have product rules of the form

$$[\lambda_l, \lambda_k] = \sum_m^{1,8} f_{lkm}\lambda_m \qquad l, k = 1, \dots 8$$

where f_{lkm} is the completely antisymmetric tensor given in [3]. f_{lkm} defines the structure constants of SU_3, (see sect. 1.7), which is then the compact group associated to A_2.

d) Calculation of roots

We outline here a method of calculating the roots of a simple Lie algebra based on properties IX, X of simple roots.

If α is a positive root we will say that α lies in the n^{th} level when:

$$\alpha = \sum_i k_i \alpha^{(i)}, \quad n = \sum_i k_i.$$

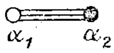
Property X) makes us sure that any root of the n^{th} level is obtained by adding a simple root to some positive root belonging to the $(n-1)^{\text{th}}$ level. In particular if the n^{th} level is empty, all the successive levels are also empty.

Suppose we know all roots up to the n^{th} level, and let $\alpha = \sum_j k_j \alpha^{(j)}$ belong to such level. Then we can ascertain whether $\alpha - l\alpha^{(k)}$, for any non negative integer l, ($\alpha^{(k)}$ is a simple root) is a root or not, so that we know the number r, relative to the $\alpha^{(k)}$-string containing α. Furthermore:

$$r - q = \frac{2(\alpha, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})} = \sum_i k_i \frac{2(\alpha^{(i)}, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})}$$

and the right hand side of this equation is a known number $(2(\alpha^{(i)}, \alpha^{(k)})/(\alpha^{(k)}, \alpha^{(k)}))$ is known from Dynkin diagram). In this way we obtain q, and if $q > 0$, $\alpha + \alpha^{(k)}$ is a root of the $(n+1)^{\text{th}}$ level. With this procedure, by (X), varying α and $\alpha^{(k)}$ we obtain all the $(n+1)^{\text{th}}$ level roots. Since we already know the roots of the 1^{th} level from Dynkin's diagram (i.e. the simple roots) and in addition $\alpha^{(i)} - \alpha^{(k)}$ is never a root when $\alpha^{(i)}$ and $\alpha^{(k)}$ are simple, this method can be used as a recurrence procedure to find all positive roots of the given algebra.

Let us try with G_2. The Dynkin diagram is:



and from (15) we obtain:

$$\frac{2(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = -1, \quad \frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = -3.$$

1^{st} level: α_1, α_2

2^{nd} level: $\alpha_1 + \alpha_2$

3^{rd} level: $2\alpha_1 + \alpha_2$ is not a root, because:

$$\frac{2(\alpha_1 + \alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = 2 - 1 = r - q;$$

but $r = 1$, so that $q = 0$.

$$\alpha_1 + 2\alpha_2 \quad \text{is a root.}$$

4^{th} level: $2\alpha_1 + 2\alpha_2 = 2(\alpha_1 + \alpha_2)$ is not a root (by V).

$\alpha_1 + 3\alpha_2$ is a root

5^{th} level: $2\alpha_1 + 3\alpha_2$ is a root: in fact $\dfrac{2(\alpha_1 + 3\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = -1 = r - q$

and r $= 0$, so that $q = 1$

$\alpha_1 + 4\alpha_2$ is not a root: $\dfrac{2(\alpha_1 + 3\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = 3 = r - q$

but $r = 3$ so that $q = 0$.

6^{th} level: $3\alpha_1 + 3\alpha_2 = 3(\alpha_1 + \alpha_2)$ is not a root

$2\alpha_1 + 4\alpha_2 = 2(\alpha_1 + 2\alpha_2)$ is not a root;

so that we end with 5^{th} level, and the positive roots are

$$\alpha_1;\ \alpha_2;\ \alpha_1 + \alpha_2;\ \alpha_1 + 2\alpha_2;\ \alpha_1 + 3\alpha_2;\ 2\alpha_1 + 3\alpha_2.$$

The corresponding root diagram is reported in fig. 1.

8.8. Compact groups associated to classical simple Lie algebras:

Cartan's denomination of $\mathscr{L}$	Compact group G associated to L	Dimension of G = dimension of $\mathscr{L}$
A_l	SU_{l+1}: unitary unimodular complex matrices in $(l+1)$-dimensions	$l(l+2)$
B_l	O_{2l+1}: real orthogonal group in $(2l+1)$ dimensions	$l(2l+1)$
C_l	$Sp(2l)$: unitary $2l$-dimensional matrices leaving invariant a non singular antisymmetric matrix I: $U^T I U = I$ (sympletic group)	$l(2l+1)$
D_l	O_{2l}: real orthogonal group in $2l$ dimensions	$l(2l-1)$

9. Representations of Semisimple Lie Algebras

9.1. We recall here that by representation of a Lie algebra into a complex linear space L we mean a linear mapping $x \to T(x)$ where $x \in \mathscr{L}$, $T(x)$ is a linear operator in L, satisfying the condition:

$$T([x, y]) = T(x)\,T(y) - T(y)\,T(x).$$

We will treat here only finite dimensional representations, for which the following Weyl's theorem applies:
I. Any finite-dimensional representation of a semisimple Lie algebra is completely reducible. Hence we can limit ourselves to irreducible representations.
Chosen a basis $\{h_i, e_\alpha, e_{-\alpha}\}$ in $\mathscr{L}$, we will indicate with $\{H_i, E_\alpha, E_{-\alpha}\}$ the corresponding operators in any given representation.
II. It is possible to choose among equivalent representation, a particular one in a Hilbert space, in which: $H_i^+ = H_i$ and $E_\alpha^+ = E_{-\alpha}$[16]).

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[16]) A representation of $\mathscr{L}$ gives us a representation of the associated compact real Lie algebra, which in turn generates a representation of the corresponding compact group. Call it $W(g)$. From what we said in sect. 3.4 we can always change $W(g)$ by an equivalence transformation $(W(g) \to W'(g) = A\,W(g)\,A^{-1})$ so to obtain an unitary representation.
Under the same equivalence transformation the operators F_i, F_α, G_α representing the compact basis (16) go into the operators:

$$F_i' = A\,F_i A^{-1} \quad \text{etc.}$$

which are antihermitian, so that H_i, E_α, $E_{-\alpha}$ transform into operators satysfying:

$$(H_i')^+ = H_i', \; (E_\alpha')^+ = E_{-\alpha}'.$$

This result is not essential from a mathematical point of view, in that what really matters is the possibility of diagonalizing the operators H_i's which is assured by the fact that H_i's represent a Cartan subalgebra. (Continued on page 334.)

Then the H_i's, being commuting Hermitian operators, are simultaneously diagonalizable, and have real eigenvalues. If $M \equiv (M_1, \dots M_r)$ is a set of eigenvalues on a simultaneous eigenvector

$$H_i |M\rangle = M_i |M\rangle$$

M can be thought as an r-dimensional real vector (weight vector) by analogy with roots. Calling L_M the manifold spanned by eigenvectors belonging to the weight M, we have

$$L = \bigoplus L_M \text{ (the direct sum runs over all weights)}. \quad (1)$$

The L_M's, in general, are not one-dimensional, so that H_i's do not constitute a complete set of commuting operators. Hence some of the weights M can be degenerate.

III. No general prescriptions can be given to construct the operators commuting with the H_i's which remove this degeneracy.

However it can be shown [*20*] that their number is at most equal to

$$\eta = \frac{n - 3r}{2} \text{ [17)]} \qquad \begin{array}{l} n = \text{dimension of } \mathscr{L} \\ r = \text{rank of } \mathscr{L} \end{array}$$

9.2. Properties of weights

Let $|M\rangle$ be a vector belonging to L_M, then by the commutation relations between H_i and E_α, we obtain:

$$H_i E_\alpha |M\rangle = (\alpha_i + M_i) E_\alpha |M\rangle . \quad (2)$$

Let us suppose $E_\alpha |M\rangle \neq 0$. Then (2) tells us that $M + \alpha \equiv (M_1 + \alpha_1 \dots M_r + \alpha_r)$ is a weight, and $E_\alpha |M\rangle$ belongs to $L_{M+\alpha}$. If $E_\alpha E_\alpha |M\rangle \neq 0$ we can repeat the reasoning concluding that $M + 2\alpha$ is a weight and that $E_\alpha E_\alpha |M\rangle$ belongs to $L_{M+2\alpha}$. By recurrence if $(E_\alpha)^k |M\rangle \neq 0$, then $M + (k\alpha)$ is a weight and $(E_\alpha)^k |M\rangle$ belongs to $L_{M+k\alpha}$. Being L finite dimensional this procedure must end, so that there exists an integer q such that $(E_\alpha)^q |M\rangle \neq 0$, i.e. $M + q\alpha$ is a weight, whereas $(E_\alpha)^{q+1} |M\rangle = 0$. By analogy we can work with $E_{-\alpha}$, obtaining an integer r such that

$$(E_{-\alpha})^r |M\rangle \neq 0$$
$$(E_{-\alpha})^{r+1} |M\rangle = 0 .$$

From this follows that all the vectors

$$M - r\alpha, \dots M, \dots, M + q\alpha \quad (3)$$

are weights, but furthermore we have:

IV. these are the only weights of the form

$$M + k\alpha \quad (k = 0, \pm 1, \pm 2, \dots).$$

However because we will use in physical applications only unitary representations of the compact group associated to L, we have adopted this particular setting from the beginning.

17) In the case of A_2, $\eta = 1$ and we will give later the explicit expression of this operator which in the physical applications is identified with the square of the isotopic spin operator.

Hence weight vectors dispose into strings generated by roots and the E_α's behave as usual raising and lowering operators.
With the aid of the tensor g^{ij} we introduce a scalar products between weights and roots, as well as between weights:

$$(M, \alpha) = \sum_{ij} g^{ij} \alpha_i M_j \tag{4}$$

$$(M, M') = \sum_{ij} g^{ij} M_i M'_j . \tag{5}$$

V. If r and q are the integers defined through (3), we have

$$\frac{2(M, \alpha)}{(\alpha, \alpha)} = r - q \tag{6}$$

hence (see 8.3, VI):

$$M - \frac{2(M, \alpha)\,\alpha}{(\alpha, \alpha)} \text{ is a weight.} \tag{7}$$

We note that the close resemblance between the stated properties of weights, and the properties of roots listed in sect. 8.3 is not surprising in that roots are simply the weights of a particular representation of $\mathcal{L}$, i.e. the regular representation.
We introduce now an ordering between the weights of an arbitrary representation. We recall that the r-simple roots constitute a basis in the space of the r-dimensional real vectors, so that for any weight M we can write:

$$M = \sum_i M_i \alpha^{(i)} \quad (\alpha^{(i)} = i^{\text{th}} \text{ simple root}). \tag{8}$$

We will say that $M > M'$ if the first non zero component of the vector $M - M'$ is greater than zero. Since there is only a finite number of distinct weights for any representation, among them there is a maximal weight, i.e. a weight which is greater than all the others. This definition has the consequence that if α is a positive root and $|\Lambda\rangle$ an eigenvector belonging to the maximal weight Λ, then $E_\alpha|\Lambda\rangle = 0$. (Otherwise $E_\alpha|\Lambda\rangle$ would be a vector belonging to the weight $\Lambda + \alpha$ which, since $\alpha > 0$, is greater than Λ).
Let R be a representation of $\mathcal{L}$ in the linear space L, and $|\Lambda, 1\rangle, |\Lambda, 2\rangle \ldots |\Lambda, k\rangle$, be independent eigenvectors belonging to the maximal weight Λ. Consider the subspace l^1 spanned by vectors

$$E_{-\alpha} E_{-\beta} E_{-\gamma} \ldots |\Lambda, 1\rangle \; (\alpha, \beta, \gamma, \ldots \text{ positive roots}) \tag{9}$$

obtained applying to $|\Lambda, 1\rangle$ all finite products of $E_{-\alpha}$'s (including repetitions of the same operators). We claim that l^1 is invariant and irreducible.
In fact it is invariant under H_i's, and $E_{-\alpha}$'s $(\alpha > 0)$ whereas applying some $E_\alpha (\alpha > 0)$ to a vector of the form (9) we can move, using commutation relations, E_α to the right, until it reaches $|\Lambda, 1\rangle$ producing zero, and leaving a combination of vectors of the form (9). (It may happen that by commuting E_α with some $E_{-\beta}$ we obtain some $E_{\alpha-\beta}$ such that $\alpha - \beta > 0$. In this case we begin to move to the right $E_{\alpha-\beta}$ until it reaches $|\Lambda, 1\rangle$). Hence when R is irreducible, $l^1 \equiv R$. In l^1 there is only one independent vector with weight Λ. In fact any eigenvector of

H_i's is a linear combination of vectors (9) differing only by the order in which E_α's appear. The corresponding weight is

$$\Lambda - k_\alpha \alpha - k_\beta \beta - k_\gamma \gamma - \cdots = \Lambda - \sum_{\alpha>0} k_\alpha \alpha \ (k_\alpha \geqq 0),$$

where k_α is the number of times $E_{-\alpha}$ appears in (9).
An eigenvector belonging to Λ is obtained when

$$\Lambda - \sum_{\alpha>0} k_\alpha \alpha = \Lambda \quad \text{i.e.} \quad \sum_{\alpha>0} k_\alpha \alpha = 0,$$

which implies $k_\alpha = 0$; hence it must be proportional to $|\Lambda, 1\rangle$.
From this it follows the irreducibility of l^1. In fact suppose l^1 to be reducible. Weyl's theorem (5.1) implies l^1 to be completely reducible. For example suppose:

$$l^1 = v_1 \oplus v_2 \ (v_1, v_2 \text{ invariant irreducible subspaces}).$$

Then:

$$|\Lambda, 1\rangle = |\Lambda, v_1\rangle + |\Lambda, v_2\rangle, \quad |\Lambda, v_r\rangle \in v_r$$

and in addition

$$H_i |\Lambda, 1\rangle = \Lambda_i |\Lambda, 1\rangle = H_i |\Lambda, v_1\rangle + H_i |\Lambda, v_2\rangle$$

i.e.

$$(H_i - \Lambda_i) |\Lambda, v_1\rangle + (H_i - \Lambda_i) |\Lambda, v_2\rangle = 0$$

$$(H_i - \Lambda_i) |\Lambda, v_1\rangle \in v_1; \ (H_i - \Lambda_i) |\Lambda, v_2\rangle \in v_2 \ \text{(by invariance of } v_1, v_2)$$

so that

$$(H_i - \Lambda_i) |\Lambda, v_1\rangle = 0$$

and the same for $|\Lambda, v_2\rangle$. Hence in l^1 there exist two independent vectors $|\Lambda, v_1\rangle$, $|\Lambda, v_2\rangle$ belonging to the weight Λ, which is impossible.
Hence $|\Lambda, v_1\rangle$ or $|\Lambda, v_2\rangle$ must vanish. Suppose $|\Lambda, v_1\rangle \neq 0$, then

$$|\Lambda, 1\rangle = |\Lambda, v_1\rangle$$

which implies $l^1 \subset v_1$ whereas by hypothesis $l^1 \supset v_1$. We conclude that $l^1 \equiv v_1$, $v_2 \equiv 0$, and l^1 is irreducible.
The great relevance of the concept of maximal weight suggested in part by previous considerations can be appreciated from the following theorems due to Cartan.
VI. Two irreducible representations having the same maximal weight are equivalent.
VII. An r-component vector Λ is the maximal weight for some irreducible representation of $\mathscr{L}$ if and only if

$$\Lambda_{\alpha_i} = \frac{2(\Lambda, \alpha^{(i)})}{(\alpha^{(i)}, \alpha^{(i)})}$$

is a non negative integer for any simple root $\alpha^{(i)}$ of $\mathscr{L}$.
Hence, once we have chosen a set of simple roots $\alpha^{(i)}$, any collection of non negative integers $(\Lambda_{\alpha_1} \Lambda_{\alpha_2}, \ldots \Lambda_{\alpha_r})$ uniquely defines an irreducible representation of $\mathscr{L}$ and all representations are obtained in this way.
VIII. If Λ is a maximal weight of a given irreducible representation of $\mathscr{L}$, then any other weight M has the form

$$M = \Lambda - \sum_i k_i \alpha^{(i)} \tag{10}$$

$$\alpha^{(i)} = i^{\text{th}} \text{ simple root}$$

$$k_i = \text{non negative integer}.$$

(The proof of this theorem easily follows from considerations preceeding result VI, and from property 8.4 X of simple roots).

(10) is quite analogous to X sect. 8.4 and a method similar to that devised in sect. 8.7 d can be based on it in order to construct all the weights of a given representation.

We dispose weights into levels according to the value of $\sum_i k_i$. Λ is assigned to the zero level.

A result analogous to sect. 8.4 XI holds, i.e.: if $M \neq \Lambda$ is a weight then there is at least one simple root such that $M + \alpha$ is a weight. This assures us that all the weights of the $(n+1)^{\text{th}}$ level are obtained subtracting some simple root to some weight of the n^{th} level and that when we reach an empty level, all the successive ones are unoccupied. Now suppose we know all the weights up to the n^{th} level and M be a weight belonging to such level. Then $M - \alpha^{(k)}$ is a weight if the integer r relative to the $\alpha^{(k)}$-string containing M is greater than zero. Moreover

$$r - q = \frac{2(M, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})} = \Lambda_{\alpha_k} - \sum_i k_i \frac{2(\alpha^{(i)}, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})}.$$

Now q is a known number (because we know all the weights up to the n^{th} level) and so is the right hand side of this equation, so that we can ascertain whether $M - \alpha^{(k)}$ is a weight or not. By varying M and $\alpha^{(k)}$ we obtain all the weights belonging to the $(n+1)^{\text{th}}$ level.

Hence, starting from the zero level, i.e. from the maximal weight (there are no negative levels because $\Lambda + \alpha^{(k)}$ is not a weight for any $\alpha^{(k)}$) by this recurrence method we can construct all the weights of the representation. This method does not provide for each weight M the corresponding multiplicity, i.e. the dimensionality of the manifold L_M appearing in (1). Being the representation determined, up to an equivalence, by its maximal weight Λ, these multiplicities must be derivable from Λ, M and from the roots, but no simple formula can be given for them. Instead we will give later for SU_3 a simple rule which allows one to read directly these multiplicities from the weight diagram.

9.2.a) Till now we have analyzed properties which are the same for equivalent representations (in particular weight diagrams).

In practical calculations it is necessary to pick-up from each equivalence class a particular representative, i.e. standard matrix representation of the elements h_i, e_α, $e_{-\alpha}$. It is convenient to choose a representation in which a basis is constituted by normalized eigenvectors of H_i's as well as of the other commuting operators which are necessary to remove degeneracies (sect. 9.1). Of course with this choice the H_i's are represented by diagonal matrices, with coefficients determined by the weight diagram.

Using the commutation relations and the string property of weights, one can determine the matrix elements of E_α, $E_{-\alpha}$ up to certain phase factors. The procedure is quite analogous, although considerably more complicated, to that employed in usual angular momentum theory [8]. It is then necessary to make a definite phase convention. At the same time this convention fixes phases in the Clebsch-Gordan coefficients.

9.3.a) Weight diagram of SU_2

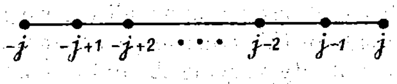
The Lie algebra of SU_2 has rank one, so that its irreducible representations are characterized by a single non negative integer Λ_{α_1}. For any maximal weight Λ, we have:

$$\frac{2(\Lambda, \alpha^{(1)})}{(\alpha^{(1)}, \alpha^{(1)})} = \Lambda_{\alpha_1} = \frac{2\,\frac{1}{g}\,\alpha_1 \Lambda_1}{\frac{1}{g}\,\alpha_1 \alpha_1} = \frac{2\Lambda_1}{\alpha_1}\,.$$

Since $\alpha_1 = 1$ (sect. 8.7a) we see that Λ_1 i.e. the covariant component of the maximal weight along $\alpha^{(1)}$ is an integer or half integer number which we call j. There is only one string i.e. the $\alpha^{(1)}$-string containing Λ, so that by applying $E_{-\alpha^{(1)}}$ to the eigenvector belonging to Λ, we generate all the vector space of the representation. There are $2j+1$ independent vectors at all, i.e.

$$|\Lambda\rangle,\ E_{-\alpha^{(1)}}|\Lambda\rangle,\ (E_{-\alpha^{(1)}})^2|\Lambda\rangle, \ldots, (E_{-\alpha^{(1)}})^{2j}|\Lambda\rangle;\ (E_{-\alpha^{(1)}})^{2j+1}|\Lambda\rangle = 0$$

so that an irreducible representation with $\Lambda_{\alpha_1} = 2j$ is $(2j+1)$-dimensional. The weight diagram is one dimensional and has the form:



so that the eigenvalues of $h_3 = iI_3$ range from j to $-j$.

9.3.b) Weight diagrams of SU_3

Being SU_3 of rank two, its irreducible representations can be labeled by two non negative integers (m, n) where, with the notations of fig. 2, we have

$$m = \frac{2(\Lambda, \alpha_2)}{(\alpha_2, \alpha_2)} \qquad n = \frac{2(\Lambda, \alpha_3)}{(\alpha_3, \alpha_3)} \tag{11}$$

and the corresponding weight diagrams are two-dimensional.
Now α_2, α_3 are linearly independent, so that we can write

$$\Lambda = n_2\alpha_2 + n_3\alpha_3$$

and, by taking the scalar product of both members with α_2 and with α_3, we obtain

$$m = 2n_2 - n_3$$

$$n = -n_2 + 2n_3$$

i.e.

$$n_2 = \frac{2m+n}{3} \qquad n_3 = \frac{m+2n}{3}\,.$$

Hence with respect to the basis chosen for roots in sect. 8.7c) we have

$$\Lambda \equiv \left(\frac{m+n}{2\sqrt{3}}\,;\ \frac{m-n}{6}\right). \tag{12}$$

These components are respectively the eigenvalues of H_1 and H_2 on the manifold L_Λ. We will construct now the weight diagram for an arbitrary irreducible representation (m, n) with a graphical method which lead to the resultmore quickly than the general one, outlined in sect. 9.2.

i) We draw with respect to two orthogonal axes the vector Λ of components $(m + n/2\sqrt{3}, m - n/6)$ as well as the two simple roots α_2, α_3 of components

$$(1/2\sqrt{3}, 1/2); (1/2\sqrt{3}, -1/2).$$

(see fig. 3).

The α_2-string containing Λ consists of the $m + 1$ weights

$$M_i = \Lambda - i\alpha_2; 0 \leqq i \leqq m.$$

The end point of M_i is obtained by reporting i times the vector $-\alpha_2$ starting from the end point of Λ. All these points lie on the segment b, spacing between two consecutive points being equal to $|\alpha_2|$, so that the lenght of b is equal to $m|\alpha_2|$. By considering the α_3-string containing Λ we obtain the segment a, of lenght $n|\alpha_3| = n|\alpha_3|$ in an analogous way.

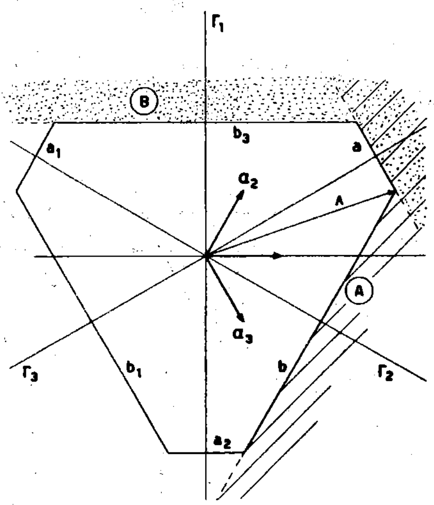


Fig. 3.

ii) There are no weights ending in the dashed region A: in fact all the weights must be of the form

$$M = \Lambda - k\alpha_2 - h\alpha_3, \quad k, h \geqq 0.$$

iii) From (7) we see that if α is a root, by reflecting a weight through an axis orthogonal to α we obtain another weight, so that the weight diagram goes into itself by such operation.

Let us indicate with r_1, r_2, r_3 the axes through the origin orthogonal to $\alpha_1, \alpha_2, \alpha_3$. Reflection through r_3 must carry a into itself (so that r_3 intersects a in its midpoint) and send b into b_3. Hence in the region B there are no weights (by reflecting such weights through r_3 we would obtain weights ending in A). Furthermore b_3 contains the end points of new weights which are the reflected of those ending in b.

Segments a_2, a_1, b_1 are obtained by reflections through r_2 and r_1. The figure so obtained has the properties that no weight ends outside it and there are weights ending on its vertices and on its sides, distances between two consecutive end points being equal to $|\alpha_1| = |\alpha_2|$.

Let us see now how we can find all the weights of the diagram.

Consider an arbitrary weight $M = \Lambda - i\alpha_2 - h\alpha_3$ (i, h non negative integers). When $i \leqq m, \Lambda - i\alpha_2$ is a weight ending on b, so that all weights M with $i \leqq m$ can be obtained from strings starting from weights ending on b. When $i \geqq m$, we write

$$M = \Lambda - m\alpha_2 - (i - m)(\alpha_2 + \alpha_2) - [h - (i - m)]\alpha_3,$$

and now, being

$$0 \leq i - m \leq n$$

the vector

$$M_{i-m} = \Lambda - m\alpha_2 - (i - m)(\alpha_2 + \alpha_3)$$

is a weight ending on a_2, so that M belongs to the α_3 string containing M_{i-m}. Hence the α_3 strings of the weights ending on b and a_2 generate the whole diagram,

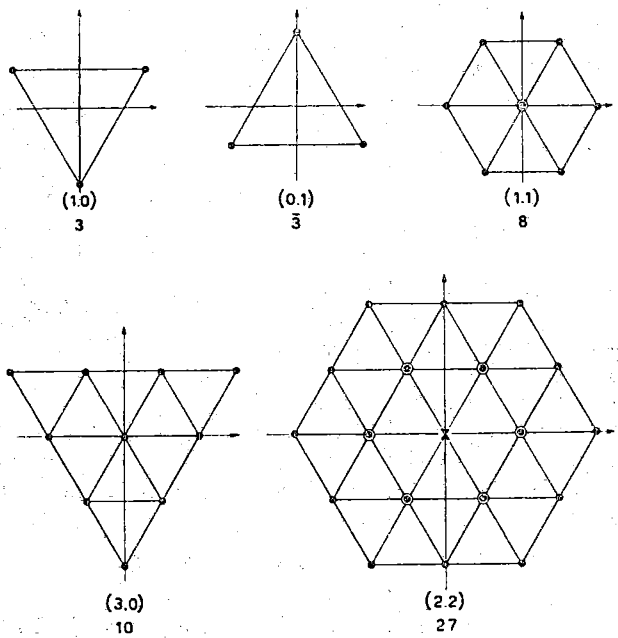


Fig 4. Weight diagrams of some SU_3 representations

and it is easy to see that the end points of the weights so obtained fill up the whole hexagon with a triangular pattern as shown in fig. 4. In addition this figure shows that the points are disposed in layers. Now the following rule applies:
Weights of the external layer are all simple, i.e. not degenerate.
Starting from the external layer, multiplicity increases by one at each layer, until a triangular path is reached: when this occurs multiplicity does not increase any more. When $n = 0$ or $m = 0$ the hexagon degenerates in a triangle and all the weights are simple.
When $n = m$ the diagram is a regular hexagon and multiplicity increases until one reach the last layer, which is constituted by a single point. In fig. 4 are given the weight diagrams of some between the most used representations of SU_3, with the respective multiplicities.
We said before that in any irreducible representation of SU_2, there is one operator, commuting with H_1 and H_2, which removes all the degeneracies.
It can be seen that the operator

$$T^2 = 3(H_1^2 + E_1E_{-1} + E_{-1}E_1) \tag{13}$$

has just these properties. We note that

$$T^2 = H_1'^2 + E_1' E_{-1}' + E_{-1}' E_1' \qquad \begin{aligned} H_1' &= \sqrt{3} H_1 \\ E_{\pm 1}' &= \sqrt{3} E_{\pm 1} \end{aligned}$$

where H_1', E_1', E_{-1}' satisfy the same commutation relations as the basis elements of a representation of A_1 (in fact they correspond to the elements h_1', e_1', e_{-1}' defined in sect. 8.7b). In any irreducible representation (m, n) of SU_3, the operators H_1', E_1', E_{-1}' generate by themselves a representation of SU_2, in general reducible. The irreducible subrepresentations of SU_2 inside (m, n) are characterized by the eigenvalue $T(T+1)$ of $T^2 (T =$ integer or half integer number). Considering beside T^2 the operator $2H_2$, one can show that in any (m, n) representation, for any pair of integers f, g such that

$$m + n \geqq f \geqq m \geqq g \geqq 0$$

there is exactly one SU_2 subrepresentation with

$$T = \frac{f-g}{2}, \qquad 2H_2 = f + g - \frac{2}{3}(m + 2n)$$

(Weyl's branching law).
For example in the (1, 1) representation there are:

1 submultiplet with	$T = 1$	$2H_2 = 0$
1 ,, ,,	$T = 1/2$	$2H_2 = 1$
1 ,, ,,	$T = 1/2$	$2H_2 = -1$
1 ,, ,,	$T = 0$	$2H_2 = 0.$

We shall see later that in the eightfold way model the operators H_1', T^2, H_2' are identified with T_3' T^2, Y, so that this rule gives us a decomposition of each SU_3 supermultiplet into isospin multiplets.

9.4. Tensor product of representations

Let ϱ_1 and ϱ_2 be two irreducible representations of $\mathscr{L}$ into the linear spaces L_1 and L_2:

$$x \in \mathscr{L} \quad x \to \varrho_1(x); \ \varrho_1(x) = \text{linear operator in } L_1$$
$$x \to \varrho_2(x); \ \varrho_2(x) = \text{linear operator in } L_2$$

and let $M^1, M^2, \ldots; N^1, N^2, \ldots;$ be the weights of the two representations, $|M^1\rangle, |M^2\rangle, \ldots |N^1\rangle, |N^2\rangle, \ldots$ the bases formed with the corresponding eigenvectors (for sake of simplicity we do not write explicitly the eigenvalues of the additional operators needed to remove all the degeneracies: they are however understood):

$$\varrho_1(hi)\,|M^k\rangle = H_i^{(1)}\,|M^k\rangle = M_i^k\,|M^k\rangle \tag{14}$$
$$\varrho_2(hi)\,|N^l\rangle = H_i^{(2)}\,|N^l\rangle = N_i^l\, N^l\rangle .$$

Then in the tensor product space $L = L_1 \otimes L_2$ which is spanned by the basis

$$|M^k\rangle\,|N^l\rangle$$

the tensor product representation ϱ of $\mathcal{L}$ is defined as

$$x \to (\varrho_1 \otimes \varrho_2)(x) = \varrho(x) \quad \varrho(x) = \text{linear operator in } L$$

$$(\varrho_1 \otimes \varrho_3)(x) \; |M^k\rangle\,|N^l\rangle = \varrho(x)\,|M^k\rangle\,|N^l\rangle = (\varrho_1(x)\,|M^k\rangle)\,|N^l\rangle + |M^k\rangle\,(\varrho_2(x)\,|N^l\rangle).$$

Recalling the definition of tensor product of two operators given in sect. 5.1 we see that

$$\varrho(x) = \varrho_1(x) \otimes 1^{(2)} + 1^{(1)} \otimes \varrho_2(x) \tag{15}$$

where $1^{(i)}$ is the identity operator in L_i. With this definition, the representation of the compact Lie group associated to $\mathcal{L}$ which is generated by ϱ, is just the tensor product of the representations of the same group generated by ϱ_1 and ϱ_2, as defined in sect. 5.1.

For the elements $\varrho(h_i)$ we have

$$\varrho(h_i) = H_i^{(1)} \otimes 1^{(1)} + 1^{(1)} \otimes H_i^{(2)}$$

so that

$$\varrho(h_i)\,|M^k\rangle\,|N^l\rangle = (M_i^k + N_i^l)\,|M^k\rangle\,|N^l\rangle,$$

i.e. the weights of ϱ are obtained by adding together the weights of ϱ_1 and ϱ_2 in all possible ways. In particular for the greatest weight we have

$$\Lambda = \Lambda^1 + \Lambda^2$$

$$\Lambda_{\alpha_i} = \Lambda^1_{\alpha i} + \Lambda^2_{\alpha i},$$

(for any simple root $\alpha^{(i)}$) where Λ^1 and Λ^2 are the maximal weights of ϱ_1 and ϱ_2. Moreover the eigenspace corresponding to Λ is always one-dimensional, and it is spanned by the vector

$$|\Lambda^1\rangle\,|\Lambda^2\rangle.$$

In general ϱ splits up in a direct sum of irreducible components

$$\varrho = \bigoplus_{\Lambda'} \varrho_{\Lambda} \,. \tag{16}$$

Each of them, according to VI, is characterized by its maximal weight Λ', which we have chosen as a label in (16). In general in this formula will appear many irreducible equivalent components, i.e. terms with the same Λ'. Although general formulae can be given, characterizing which are the irreducible components and how many times they appear in (16) [*18*] yet these formulae are extremely complicated[18]) and are not used in practice[19]).

---

[18]) This is not the case of $A_1(SU_2)$ for which the decomposition (16) is explicity given by the well know Clebsch-Gordan formula:

$$\varrho = (\varrho_{j_1} \otimes \varrho_{j_2}) = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \varrho_j \,.$$

[19]) In practical calculations we need not only the decomposition (16), but also explicitly the matrix connecting the basis $|M^k\rangle\;|N^l\rangle$ to the basis spanning the irreducible components, i.e. we need all the Clebsch-Gordan coefficients involved. Such coefficients in general have not been calculated; however in the case of SU_3 they are tabulated [*29, 33*] for all tensor product of interest in physics.

Anyway it is a very simple task to isolate a particular term in (16), i.e. the component ϱ_Λ where Λ is the greatest weight of ϱ. In fact if we construct the manifold spanned by the vectors

$$E_{-\alpha} E_{-\beta} E_{-\gamma} \dots |\Lambda\rangle$$

($\alpha, \beta, \gamma, \dots$ = positive roots: $E_{-\alpha}, E_{-\beta}, E_{-\gamma}, \dots$ lowering operators of ϱ) we obtain an irreducible invariant subspace (see sect. 9.2) and obviously if we consider the restriction of ϱ to this manifold, we obtain a representation having the maximal weight equal to Λ. Being Λ simple, ϱ_Λ occurs only once in (16).
Consider now the r-irreducible inequivalent representations ϱ_i with maximal weights $\Lambda^{(i)}$, such that

$$\frac{2(\Lambda^{(i)}, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})} = \delta_{ik}.$$

Then we have: any irreducible representation ϱ identified by the set of non negative integers $(\Lambda_{\alpha_1}, \Lambda_{\alpha_2}, \dots \Lambda_{\alpha_r})$ can be obtained by making the tensor product

$$\underbrace{\varrho_1 \otimes \varrho_1 \cdots}_{\Lambda_{\alpha_1}\text{ terms}} \quad \underbrace{\otimes\, \varrho_2 \otimes \varrho_2 \cdots}_{\Lambda_{\alpha_2}\text{ terms}} \quad \underbrace{\otimes\, \varrho_2 \otimes \varrho_2 \cdots}_{\Lambda_{\alpha_r}\text{ terms}}, \tag{17}$$

and isolating the irreducible component of greatest weight. In fact this component belongs to the weight

$$\Lambda = \Lambda_{\alpha_1} \Lambda^{(1)} + \Lambda_{\alpha_2} \Lambda^{(2)} + \cdots + \Lambda_{\alpha_r} \Lambda^{(r)}$$

which is just the maximal weight of ϱ.

9.5. Contragradient representation

Given a representation ϱ of $\mathcal{L}$ in a linear space L, we can construct another representation $\bar{\varrho}$ which is called the contragradient (or adjoint) of ϱ.
We first fix in L a basis in which the operators $\varrho(x)$ are represented by certain matrices $(\varrho(x))^i_j$; then we consider a linear space L^* having the same dimensionality of L. The representation $\bar{\varrho}$ in L^* is constituted by the operators $\bar{\varrho}(x)$ which, with respect to a basis fixed in L^*, are represented by the matrices

$$(\bar{\varrho}(x))^i_j = -(\varrho(x))^j_i = -[\varrho(x)^T]^i_j. \tag{18}$$

Operators $\bar{\varrho}$ defined in (18) will be symbolically written as

$$\bar{\varrho} = -\varrho^T.$$

For the elements of $\mathcal{L}$: $h_i, e_\alpha, e_{-\alpha}$, we have

$$\left.\begin{aligned} h_i &\to \varrho(h_i) = H_i \\ e_\alpha &\to \varrho(e_\alpha) = E_\alpha \end{aligned}\right\}\text{ for the representation } \varrho$$

$$\left.\begin{aligned} h_i &\to \bar{\varrho}(h_i) = -H_i^T \\ e_\alpha &\to \bar{\varrho}(e_\alpha) = -E_\alpha^T \end{aligned}\right\}\text{ for the representation } \bar{\varrho},$$

so that the eigenvalue of $\bar{\varrho}(h_i)$ are just minus one times the eigenvalues of $\varrho(h_i)$, i.e. the weight diagram of $\bar{\varrho}$ is obtained by reflecting through the origin the diagram of ϱ. In particular $\bar{\varrho}$ is equivalent to ϱ if and only if its weight diagram is invariant under this reflection.

Consider a vector $x \in L$ with components (x^i) and a vector $y \in L^*$ with components (y_i): by definition applying transformations ϱ and $\bar{\varrho}$, we have

$$x'^i = \sum_k \varrho^i_k x^k$$

$$y'_i = \sum_k - \varrho^k_i y_k .$$

Having L and L^* the same dimension, there exists always a one-to-one correspondence between their elements. Let

$$x = Ay \quad x \in L, y \in L^*$$

be such correspondence. If we transform x with ϱ and y with $\bar{\varrho}$, the resulting vectors will not be connected by A, unless ϱ is equivalent to $\bar{\varrho}$

$$x' = \varrho x, \; y' = \bar{\varrho} y, \quad \text{then}$$

$$x' = Ay', \quad \text{implies}$$

$$\varrho = A\bar{\varrho}A^{-1}.$$

Hence only if $\varrho \sim \bar{\varrho}$ we can identify L and L^* in a way which is invariant under ϱ and $\bar{\varrho}$.

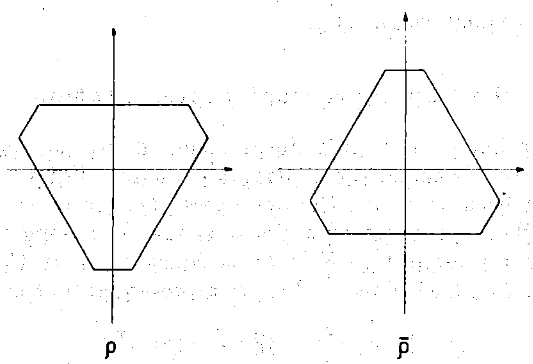


Fig. 5.

We note that the definition of adjoint representation has its analogous if we consider the group associated with $\mathscr{L}$, in that, if

$$g \to T(g)$$

is the representation of this group generated by ϱ, then

$$g \to \bar{T}(g) = (T(g)^{-1})^T$$

is that generated by $\bar{\varrho}$.

In the case of SU_3 we note that all weights diagrams are symmetrical for reflections around y-axis (this axis is in fact orthogonal to the root α_1) so that, to obtain the diagram of the adjoint representation, we have only to reverse it with respect to the x-axis.

A glance to figure 5 is sufficient to conclude that if ϱ is the representation (m, n), then $\bar{\varrho}$ is (n, m). In particular ϱ is equivalent to $\bar{\varrho}$ if and only if $m = n$ (in this case in fact they have the same maximal weight).

9.6. Explicit construction of SU_3 representations

The 3×3 antihermitian traceless matrices

$$\lambda_1 = -\frac{i}{2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \lambda_2 = -\frac{i}{2}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \lambda_3 = -\frac{i}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\lambda_4 = -\frac{i}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix} \quad \lambda_5 = -\frac{i}{2}\begin{pmatrix}0 & 0 & i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix} \quad \lambda_6 = -\frac{i}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \tag{19}$$

$$\lambda_7 = -\frac{i}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix} \quad \lambda_8 = -\frac{i}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}$$

satisfy the commutation relations given in sect. 8.7c so that they are a 3-dimensional representation of the compact basis of A_2, and their real combinations span a representation (in fact irreducible) of the Lie algebra of SU_3. The operators representing h_1 and h_2, are (see sect. 8.7c)

$$H_1 = \frac{i\lambda_3}{\sqrt{3}} = \frac{1}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$H_2 = \frac{i\lambda_8}{\sqrt{3}} = \frac{1}{6}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}.$$

From these we obtain three weights

$$\Lambda^1 = \left(\frac{1}{2\sqrt{3}}, \frac{1}{6}\right); \quad \Lambda^2 = \left(-\frac{1}{2\sqrt{3}}, \frac{1}{6}\right); \quad \Lambda^3 = \left(0, -\frac{1}{3}\right).$$

By considering the simple roots α_2 and α_3 we easily see that Λ^1 is the greatest weight and that

$$\frac{2(\Lambda^1, \alpha_2)}{(\alpha_2, \alpha_2)} = 1 \qquad \frac{2(\Lambda^1, \alpha_3)}{(\alpha_3, \alpha_3)} = 0,$$

so that this representation is the (1,0) one, whose diagram is reported in fig. 4.

The adjoint representation is obtained by taking the matrices $-\Lambda_i^T$, and is the (0,1) representation, inequivalent to the (1,0) one. These representations which are the fundamental ones, being the only 3-dimensional representations of A_2 will be called 3 and $\overline{3}$.
By taking the matrices

$$T(\alpha_1 \ldots \alpha_8) = e^{\sum_k^{1,8} \alpha_k \lambda_k} \qquad (\alpha_k \text{ real numbers}),$$

we obtain the set of 3×3 unitary unimodular matrices which constitute a group of operators in a 3-dimensional complex space. This group is by definition SU_3.
Representation $\overline{3}$ generates the operators

$$\overline{T}(\alpha_1 \ldots \alpha_1) = e^{-\sum_k \alpha_k \lambda_k^T} = \overline{\left(e^{\sum_k \alpha_k \lambda_k}\right)}.$$

This is the representation which to each matrix of SU_3 associates the complex conjugate matrix.
The linear spaces of representations 3 and $\overline{3}$ are

L_3: general vector x indicated as (x^1, x^2, x^3)

L_3^*: ,, ,, y ,, ,, (y_1, y_2, y_3).

If U_{ik} is a matrix of SU_3, then for the representation 3 we have

$$U \to T(U): x' = T(U)x$$
$$x'^i = \sum_k U_{ik} x^k,$$

whereas for $\overline{3}$

$$U \to \overline{T}(U), \quad y' = \overline{T}(U)\,y$$
$$y'_i = \sum_k \overline{U}_{ik}\, y_k.$$

The tensor product

$$L_n^m = \underbrace{L_3 \otimes L_3 \otimes \ldots L_3}_{m} \otimes \underbrace{L_3^* \otimes L_3^* \ldots \otimes L_3^*}_{n}$$

is just the vector space of the 3^{m+n} components objects (tensors) (sect. 5.1)

$$X^{i_1 \ldots i_m}_{j_1 \ldots j_n} \qquad (i_1, \ldots i_m; j_1, \ldots j_n = 1,2,3),$$

and the product representation of SU_3 in this space is constituted by the operators $T^{(m)}_{(n)}(U)$ defined as (sum over repeated indices is understood)

$$(T^{(m)}_{(n)}(U)X)^{i_1 \ldots i_m}_{j_1 \ldots j_n} = U_{i_1 i_1'}\, U_{i_2 i_2'} \ldots U_{i_m i_m'}\, \overline{U}_{j_1 j_1'} \ldots \overline{U}_{j_n j_n'}\, X^{i_1' \ldots i_m'}_{j_1' \ldots j_n'}. \qquad (20)$$

From (15) sect. 9.4 we see that the compact basis of the Lie algebra of SU_3 is represented by the operators $\Lambda^{(k)}$ acting as

$$\begin{aligned}(\Lambda^{(k)}X)^{i_1\ldots i_m}_{j_1\ldots j_n} = (\lambda^{(k)}_{i_1i'_1}\delta_{i_2i'_2}\ldots\delta_{i_mi_m'}\delta_{j_1j'_1}\ldots\delta_{j_nj_n'} + \delta_{i_1i'_1}\lambda^{(k)}_{i_2i'_2}\times\\ \times\ \delta_{i_3i'_3}\ldots\delta_{j_nj_n'} + \ldots - \delta_{i_1i'_1}\ldots\delta_{i_mi_m'}\lambda^{(k)}_{j'_1j_1}\times\\ \times\ \delta_{j_2j'_2}\ldots - \delta_{i_1i'_1}\ldots\delta_{j_{n-1}j_{n-1}'}\lambda^{(k)}_{j_n'j_n})\, X^{i'_1\ldots i_m'}_{j'_1\ldots j_n'},\end{aligned}$$

where $\lambda^{(k)}_{ij}$ are the matrices listed in (19).

9.7. Let us consider the tensor product representation (reducible when $k > 1$):

$$(3)^k = \underbrace{3 \otimes 3 \otimes 3 \otimes \ldots \otimes 3.}_{k}$$

Instead of considering the irreducible subspaces, we focus our attention on the operators which project on them. Suppose:

$$L_k = \bigotimes_\alpha L_\alpha \quad \alpha = \text{labels irreducible subspaces.}$$

If Y_α projects over $L_\alpha (Y^2_\alpha = Y_\alpha)$, then we have:

a) $Y_\alpha T^{(k)}(U) = T^{(k)}(U) Y_\alpha$ for any $U \in SU_3$;

b) there exists no $Y_{\alpha'}$, projecting on a subspace $L_{\alpha'} \subset L_\alpha$ commuting with $T^{(k)}(U)$'s, i.e. Y_α are minimal projections;

c) Y_α's are orthogonal ($Y_\alpha Y_\beta = 0$ when $\alpha \neq \beta$) and constitute a complete set $\left(\sum_\alpha Y_\alpha = 1\right)$.

To characterize Y_α, we have to consider the set of all operators commuting with $T^{(k)}(U)$'s.

Let us call p an arbitrary permutation of $1, 2 \ldots k$:

$$p: \quad 1 \to 1', 2 \to 2', \ldots k \to k',$$

where $1', 2' \ldots k'$ are again the numbers $1, 2 \ldots k$ rearranged in some way specified by p. We associate to p a linear operator p acting on L_k, defined as:

$$(pX)^{i_1\ldots i_k} = X^{i_{1'}\ldots i_{k'}}.$$

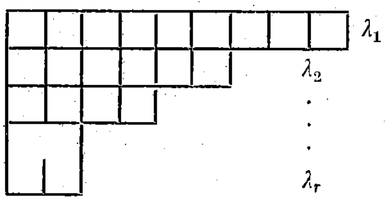
Such operators constitute of course a representation of the group of all permutations of k-objects.

Moreover:

$$\begin{aligned}(pT^{(k)}(U)X)^{i_1\ldots i_k} &= (T^{(k)}(U)X)^{i_{1'}\ldots i_{k'}} = U_{i_{1'}j_1}\ldots U_{i_{k'}j_k}X^{j_1\ldots j_k} =\\ &= U_{i_{1'}j_{1'}}\ldots U_{i_{k'}j_{k'}}X^{j_{1'}\ldots j_{k'}} = U_{i_{1'}j_{1'}}\ldots U_{i_{k'}j_{k'}}(pX)^{j_1\ldots j_k} =\\ &= U_{i_1j_1}\ldots U_{i_kj_k}(pX)^{j_1\ldots j_k} = (T^{(k)}(U)\,pX)^{i_1\ldots i_k},\end{aligned}$$

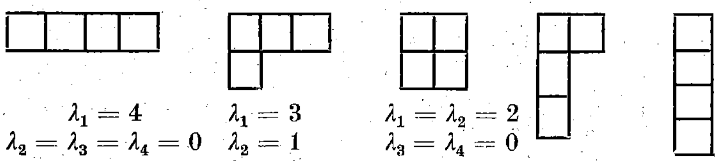
i.e. p commutes with $T^{(k)}(U)$. Let us call Σ_k the set of all the operators p and of all their linear combinations: clearly all the elements of Σ_k commute with

$T^{(k)}(U)$'s, but the importance of Σ_k lies in the fact that the converse is also true, i.e. [21]:
all the operators commuting with $T^{(k)}(U)$'s belong to Σ_k.
Hence Y_α's (which reduce the representation) are in Σ_k.
Their characterization can be achieved using the so called Young tableaux.
Consider an arbitrary partition of k objects into groups of $\lambda_1, \lambda_2 \ldots \lambda_r$ elements ($\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_r$). This partition is indicated by a tableau made of r rows containing $\lambda_1, \lambda_2, \ldots, \lambda_r$ boxes:



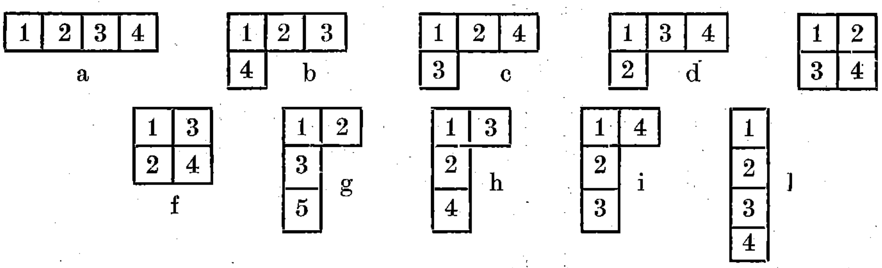
$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = k.$$

Fill now this tableau with the numbers 1, 2, ... k in all possible ways consistent with the rule: numbers must increase, in each row from left and in each column from above.
Consider for example $k = 4$. Then we have five tableaux:



$\lambda_1 = 4$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$ $\quad$ $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = \lambda_4 = 0$ $\quad$ $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = \lambda_4 = 0$ $\quad$ $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$, $\lambda_4 = 0$ $\quad$ $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$

which, filled in all possible ways, give:



(21)

To each tableau determined by a partition $\lambda_1, \ldots \lambda_r$ ($\lambda_1 + \lambda_2 + \cdots = k$) and by a particular arrangement of the numbers $1, 2 \ldots k$ consistent with previous rule, we associate an operator Y (Young symmetrizer) defined as

$$Y = QP, \tag{22}$$

P = sum of all operators associated to permutations of $1, 2 \ldots k$ which leave unchanged the rows of the tableaux $= \sum_p p$, Q = sum over all permutations q which leave columns unchanged, each being multiplied by its signature $Q = \sum_p \delta_q q$.

Let us indicate permutations (as well as operators which represent them) in the cyclic notation: for example:

$$(1\ 2\ 4)\,(3) \quad \text{or simply } (1\ 2\ 4)$$

stands for:

$$1 \to 2$$
$$2 \to 4$$
$$4 \to 1$$
$$3 \to 3$$

and:

$$(1\ 4)\,(3\ 2)$$

stands for:

$$1 \to 4$$
$$4 \to 1$$
$$2 \to 3$$
$$3 \to 2.$$

Then e.g. tableau (21) f is associated to the operator:

$$Y_f = (e - (12) - (34) + (12)\,(34))\,(e + (13) + (24) + (13)\,(24))$$

(e is the identity permutation).
Applied to the general tensor $X^{i_1i_2i_3i_4}$, this operator gives the tensor

$$X_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 & i_4 \\ \hline\end{array}} :$$

$$\begin{aligned} X_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 & i_4 \\ \hline\end{array}} = {} & X^{i_1i_2i_3i_4} + X^{i_3i_2i_1i_4} + X^{i_1i_4i_3i_2} + X^{i_3i_4i_1i_2} \\ & - X^{i_2i_1i_3i_4} - X^{i_2i_3i_1i_4} - X^{i_4i_1i_3i_2} - X^{i_4i_3i_1i_2} \\ & - X^{i_1i_2i_4i_3} - X^{i_3i_2i_4i_1} - X^{i_1i_4i_2i_3} - X^{i_3i_4i_2i_1} \\ & + X^{i_2i_1i_4i_3} + X^{i_2i_3i_4i_1} + X^{i_4i_1i_2i_3} + X^{i_4i_3i_2i_1}. \end{aligned}$$

It can be shown that:
i) all Y's are proportional to projections:

$$Y^2 = c\,Y \quad \text{so that } \frac{Y}{c} \text{ is a projection,}$$

ii) Y's are minimal projections, and

$$Y\,Y' = 0,$$

when Y' corresponds to a different tableau than Y (i.e. differing for the partition $\lambda_1, \lambda_2, \ldots \lambda_r$ or for a different arrangement of the numbers $1, 2, \ldots k$);
iii) for a given k the set of projections Y associated to all possible Young tableaux of order k completely reduces the representation $(3)^k$;

iv) Y's corresponding to the same partition $\lambda_1, \dots \lambda_r$, differing by the arrangement of $1, 2 \dots k$ project over equivalent representations.

It can be easily verified that any Young symmetrizer projects over tensors antisymmetrical in indices appearing in the same column of the corresponding tableau. Now our indices run over 1, 2, 3 so that each tableau with more than three rows gives an identically vanishing projection (this is the case of the tableau (21)l) and need not to be considered.

We want to see now how each irreducible (m, n) representation of SU_3 can be obtained isolating a particular irreducible component in a suitable product $3 \otimes 3 \dots \otimes 3$ [*21*].

Consider the tensor product $T^{(m+2n)} = (3)^{m+2n}$. We claim that the irreducible manifold L characterized by the Young tableau

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2\dots & n & & m+n \\ \hline m+n+1 & \dots & m+2n \\ \cline{1-3}\end{array} \tag{23}$$

transforms as the (m, n) representation.

Let us call $L^{(m)}_{(n)}$ the vector space of tensor with m upper and n lower indices; then the following mapping:

$$F^{i_{n+1}\dots i_{n+m}}_{j_1\dots j_n} = \varepsilon_{j_1 i_1 i_{n+m+1}}\varepsilon_{j_2 i_2 i_{n+m+2}} X_{\begin{array}{|c|c|c|c|c|c|}\hline i_1 & i_2 & \dots & i_{n+1} & & i_{n+m} \\ \hline i_{n+m+1} & i_{n+m+2} & \dots \\ \cline{1-3}\end{array}} \tag{24}$$

or, in short, $F^{(i)}_{(j)} = \varepsilon_{(j)(ke)} X_{\begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array}}$

(we contract with ε_{ijk} each pair of indices belonging to the same column) induces a one-to-one correspondence between L and a linear manifold (which we will specify later) contained in $L^{(m)}_{(n)}$. That (24) is a one-to-one mapping can be seen by showing directly that if

$$F^{(i)}_{(j)} = \varepsilon_{(i)(ke)} X_{\begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array}} = \varepsilon_{(i)(ke)} X'_{\begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array}},$$

then

$$X_{\begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array}} = X'_{\begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array}}.$$

To do this, write explicitly the expression

$$0 = \varepsilon_{j_1 i_1 i_{n+m+1}}\varepsilon_{j_2 i_2 i_{n+m+2}} \dots (X - X')_{\begin{array}{|c|c|c|c|c|c|}\hline i_1 & i_2 & \dots & i_{n+1} & \dots & i_{n+m} \\ \hline i_{n+m+1} & i_{n+m+2} & \dots \\ \cline{1-3}\end{array}},$$

then contract indices $j_1, j_2, \dots j_n$ with $\varepsilon_{s_1 t_1 j_1}, \varepsilon_{s_2 t_2 j_2}, \dots$ Using the idenuity:

$$\varepsilon_{stj}\,\varepsilon_{jii'} = \delta_{si}\,\delta_{ti'} - \delta_{si'}\,\delta_{ti}$$

as well as antisymmetry of tensor X in the indices of the same column, the wanted result is obtained.
Moreover a simple but rather tedious calculation leads to the relation

$$T^{(m)}_{(n)}(U)\,F = T^{(m)}_{(n)}\,\varepsilon X = \varepsilon\, T^{m+2n}(U)\,X,$$

where $T^{(m)}_{(n)}(U)$ is any element of the representation $(3)^m \times (\bar{3})^n = \underbrace{3 \otimes 3 \dots 3}_{m} \otimes \underbrace{\otimes\, \bar{3} \dots \otimes \bar{3}}_{n}$ and with ε we have indicated the mapping (24). This shows that the image L^ε of L into $L^{(m)}_{(n)}$ is an irreducible subspace for the representation $(3)^m \times (\bar{3})^n$, whose restriction to L^ε is equivalent to the restriction of $T^{(m+2n)}$ to L.
Proof will be complete if we show that the vector belonging to the maximal weight of $(3)^m \times (\bar{3})^n$ is contained in L^ε (see sect. 5.4).
Now, according to the matrices of sect. 9.6, the vector belonging to the maximal weight of the representation 3 is

$$(\bar{x})^i = \delta^{i1} = (1, 0, 0),$$

whereas for the representation $\bar{3}$ it is

$$(\bar{y})_i = \delta_{i2} = (0, 1, 0).$$

Hence the tensor of $L^{(m)}_{(n)}$ belonging to the maximal weight of $(3)^m \otimes (3)^n$ is

$$(\bar{F})^{i_{n+1}\dots i_{n+m}}_{j_1\dots j_n} = \delta^{i_{n+1}1}\,\delta^{i_{n+2}1}\dots\delta^{i_{n+m}1}\cdot\delta_{j_1 2}\dots\delta_{j_n 2}$$

and we have to show that $\bar{F}$ belongs to L^ε.
Consider the tensor $\bar{X}$ (belonging to L) defined as

$$\bar{X}\begin{array}{|c|c|c|c|}\hline 1 & \dots & 1 & \dots 1 \\ \hline 3 & \dots & 3 \\ \cline{1-3}\end{array} = \left(\frac{-1}{2}\right)^n$$

$$\bar{X}\begin{array}{|c|c|c|}\hline i_1 & \cdot\,\cdot & \dots\, i_{n+m} \\ \hline i_{n+m+1} & \cdot\,\cdot \\ \cline{1-2}\end{array} = 0,$$

when indices appearing in columns are not permutations of 1 and 3, and some of the last m indices in first row is different from 1. Then obviously we have

$$(\bar{F})^{(i)}_{(j)} = \varepsilon_{(j)\,(ke)}\,\bar{X}\begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array}\,.$$

i.e. $\bar{F}$ belongs to L^ε, Q.E.D.

For example previous theorem allows one to conclude that linear manifolds on which tableaux (21 b, c, d) project, transform as the (2,1) representation, whereas those corresponding to (21 e, f) transform as (0,2).
In the composition of the tensor product $(3)^k$, also tableaux with three rows appear. To which irreducible representation do they correspond? We will see this referring to an example.
Consider the manifold of tensors

$$X_{\begin{array}{|c|c|}\hline i_1 & i_4 \\ \hline i_2 \\ \cline{1-1} i_3 \\ \cline{1-1}\end{array}} .$$

belonging to the product $(3)^4$ (tableau (21 i)).
It transforms under SU_3 as

$$(T^{(4)}(U)\,X)_{\begin{array}{|c|c|}\hline i_1 & i_4 \\ \hline i_2 \\ \cline{1-1} i_3 \\ \cline{1-1}\end{array}} = U_{i_1j_1}U_{i_2j_2}U_{i_3j_3}U_{i_4j_4}\,X_{\begin{array}{|c|c|}\hline j_1 & j_4 \\ \hline j_2 \\ \cline{1-1} j_3 \\ \cline{1-1}\end{array}}$$

Being X antisymmetrical in j_1, j_2, j_3, we can write

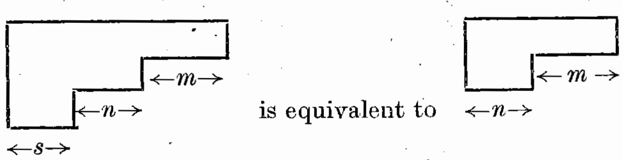
$$X_{\begin{array}{|c|c|}\hline j_1 & j_4 \\ \hline j_2 \\ \cline{1-1} j_3 \\ \cline{1-1}\end{array}} = \varepsilon_{j_1j_2j_3}\,X_{\begin{array}{|c|c|}\hline 1 & j_4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}} ,$$

i.e.

$$T^{(4)}(U)\,X_{\begin{array}{|c|c|}\hline i_1 & i_4 \\ \hline i_2 \\ \cline{1-1} i_3 \\ \cline{1-1}\end{array}} = (U_{i_1j_1}U_{i_2j_2}U_{i_3j_3}\,\varepsilon_{j_1j_2j_3})\,U_{i_4j_4}\,X_{\begin{array}{|c|c|}\hline 1 & j_4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}}$$

$$= \det U\,\varepsilon_{i_1i_2i_3}\,U_{i_4j_4}\,X_{\begin{array}{|c|c|}\hline 1 & j_4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}} = U_{i_4j_4}\,X_{\begin{array}{|c|c|}\hline i_1 & j_4 \\ \hline i_2 \\ \cline{1-1} i_3 \\ \cline{1-1}\end{array}} .$$

so that the irreducible component $(21\,i)$ is equivalent to the representation **3**, which corresponds to the tableau $\square$. This reasoning can be repeated for tableaux with an arbitrary number s of columns with three boxes, leading to the conclusion that:



Using these arguments we find, for example, the decompositions:

$$(3)^4 = \underset{a}{(4,0)} + \underbrace{(2,1) + (2,1) + (2,1)}_{bcd} + \underbrace{(0,2) + (0,2)}_{ef} + \underbrace{(1,0) + (1,0) + (1,0)}_{ghi}$$

$$(3)^2 = (0,1) + (2,0)$$

$$(3)^3 = (3,0) + (1,1) + (1,1) + (0,0).$$

A formula can be given for the dimension of the (m, n) representation [21] by counting independent tensors of a given tableau

$$d_{mn} = (1 + m)(1 + n)\left(1 + \frac{m+n}{2}\right). \tag{25}$$

Note that, when this produces no ambiguity, we will use d_{mn} to indicate the representation (m, n) $(m \geqq n)$ and $\bar{d}_{mn}$ to indicate its contragradient (n, m) (for example 10 for (3,0) and $\overline{10}$ for (0,3)).
We want now to characterize the manifold L^ε into which L is mapped by (24). More definitely we show that L^ε is the linear manifold of all tensors $F^{i_1\ldots i_m}_{j_1\ldots j_n}$ symmetrical in upper and lower indices and traceless, i.e. such that

$$F^{i\,i_2\ldots i_m}_{i\,j_2\ldots j_n} = 0. \tag{26}$$

We show this in two steps:

i) any tensor in L^ε is a linear combination of tensors with the stated properties. In fact we proved before that in L^ε there is the tensor

$$(\bar{F})^{i_1\ldots i_m}_{j_1\ldots j_n} = \delta^{i_1 1} \ldots \delta^{i_m 1}\, \delta_{j_1 2} \ldots \delta_{j_n 2},$$

which of course is symmetrical and traceless. Consider tensors

$$F(U) = T^{(m)}_{(n)}(U)\, \bar{F},$$

where U runs over the whole SU_3. We have

$$F(U)^{i_1\ldots i_m}_{j_1\ldots j_n} = U_{i_1 i_1'} \ldots U_{i_m i_m'}\, \bar{U}_{j_1 j_1'} \ldots \bar{U}_{j_n j_n'}\, \delta^{i_1' 1} \ldots \delta^{i_m' 1} \ldots \delta_{j_n' 2} =$$
$$= U_{i_1 1} \ldots U_{i_m 1}\, \bar{U}_{j_1 2} \ldots \bar{U}_{j_n 2}.$$

$F(U)$'s are of course symmetrical in upper and lower indices and, due to unitarity of the U's, they are traceless. The linear manifold spanned by $F(U)$'s is contained in L^ε and is obvously invariant for the representation $(3)^m \times (\bar{3})^n$. Since L^ε is irreducible (as we saw before) and this manifold is surely not zero, we conclude that it is identical with L^ε; i.e. L^ε contains tensors symmetrical and traceless.
ii) Call L^m_n the manifold of all tensors symmetrical in upper and lower indices and traceless. From i) $L^m_n \supset L^\varepsilon$. We show now that the dimension of L^m_n equals that of L^ε, so that $L^m_n = L^\varepsilon$.

A tensor $F^{i_1 \dots i_m}_{j_1 \dots j_n}$ symmetrical in upper and lower indices, has

$$\binom{m+3-1}{m}\binom{n+3-1}{n} = \frac{(m+2)!}{m!\,2!}\,\frac{(n+2)!}{n!\,2!} = \frac{(m+2)(m+1)(n+2)(n+1)}{4} \tag{27}$$

independent components (remember that indices i, j go over 1, 2, 3), and the number of independent conditions (26) equals the number of independent components of a tensor symmetrical in $(m-1)$ upper an $(n-1)$ lower indices, i.e. it is equal to:

$$\binom{(m-1)+3-1}{m-1}\binom{(n-1)+3-1}{n-1} = \frac{m(m+1)\,n\,(n+1)}{4}. \tag{28}$$

Substracting (28) from (27) we obtain:

$$\dim\,(L^m_n) = (m+1)\,(n+1)\left(1+\frac{m+n}{2}\right) = \dim L.$$

Because $\dim L^\varepsilon = \dim L$ being (24) a one-to-one mapping, we conclude:

$$\dim L^\varepsilon = \dim L^m_n$$

i.e.

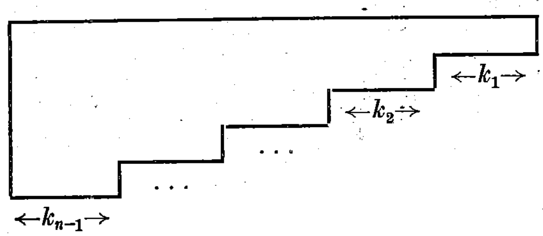
$$L^\varepsilon = L^m_n.$$

Hence we have found another possible realization of the $(m, n)SU_3$ representation: the space of tensors $F^{i_1 \dots i_m}_{j_1 \dots j_m}$ symmetrical in upper and lower indices and traceless, transforming as:

$$F^{i_1 \dots i_m}_{j_1 \dots j_n} \to (F')^{i_1 \dots i_m}_{j_1 \dots j_n} = U_{i_1 i_1'} \dots U_{i_m i_m'}\,\overline{U}_{j_1 j_1'} \dots \overline{U}_{j_n j_n'}\,F^{i_1' \dots i_m'}_{j_1' \dots j_n'}.$$

A large part of the above results can be generalized to representations of an arbitrary group SU_n. According to the general results of sect. 9.2 the irreducible SU_n representations can be labeled by $(n-1)$ positive integers $(k_1, k_2, \dots, k_{n-1})$. In particular the representation $(1, 0, 0, \dots, 0)$ is always made up with SU_n matrices themselves, and is n dimensional. Results i) ii) applied to the reduction of the tensor product $(n)^k = \underset{1}{n} \otimes \underset{2}{n} \otimes \dots \otimes \underset{k}{n}$ hold unchanged: in this case of course indices $i_1 \dots i_k$ run over $1, 2, \dots n$, and we have to consider tableaux containing up to n rows.

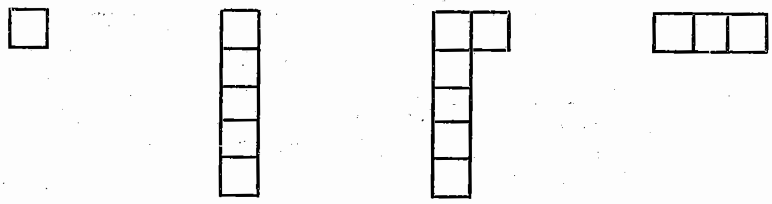
A tableau of the form



is associated to the representation $(k_1,..k_{n-1})$, and tableaux differing by an arbitrary number of columns with n rows characterize equivalent representations. The dimensionality of the representation $(k_1,..k_{n-1})$ is given by the formula:

$$d(k_1,..k_{n-1}) = \prod_{l}^{0,n-2} \prod_{s}^{1,n-l-1} \left(1 + \frac{k_s + k_{s+1} + \cdots + k_{s+l}}{l+1}\right).$$

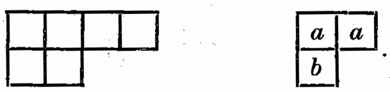
We list here for example some SU_6 representations, together with their dimensionality (the contragradient of $(k_1 \ldots k_{n-1})$ is $(k_{n-1}, k_{n-2}, .., k_1)$)



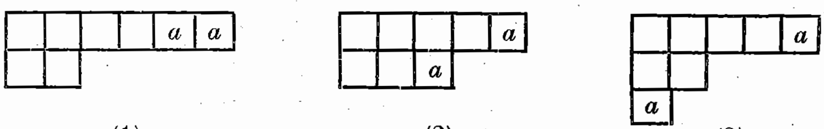
(1, 0, 0, 0, 0) (0, 0, 0, 0, 1) (1, 0, 0, 0, 1) (3, 0, 0, 0, 0)

Dim: 6 $\overline{6}$ 35 56

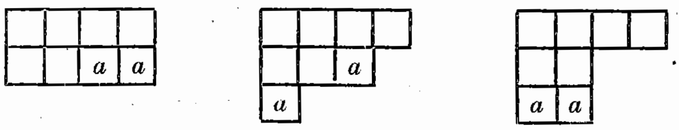
Reduction of tensor product of arbitrary SU_3 representations.
In finding the irreducible components of the product $(m, n) \otimes (m', n')$ again the technique of Young tableaux can be used. We give here without proof a simple rule for making this reduction [22]. We illustrate this rule referring to the product $(2,2) \otimes (1,1)$.
Write down the corresponding tableaux, having filled with symbols a and b first and second row of one of the two, arbitrarily selected:



Then add to the empty tableau the boxes of the first row of the second one in all possible ways consistent with the rules:
i) there must never appear two a's in the same column;
ii) for each resulting tableau, containing $\lambda_1, \lambda_2, \lambda_3$ boxes in first, second, third row, it must be $\lambda_1 \geqq \lambda_2 \geqq \lambda_3$.
In our case we have six possible tableaux:

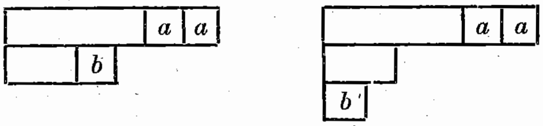


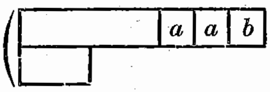
(1) (2) (3)



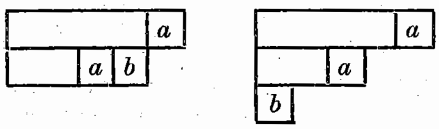
(4) (5) (6)

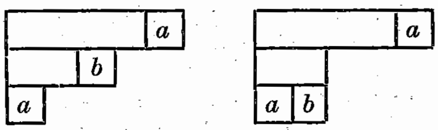
To all these tableaux we add now, in all possible ways, all boxes containing b, again consistently with ii) and with the additional rules;
iii) there must never appear two b's in the same column;
iv) for each tableau so obtained write the sequence of a's and b's obtained reading first row from right to left, then the second row from right to left and so on: only tableaux are allowed such that at any stage of this sequence the number of a's already read is greater than or equal to the corresponding number of b's.
From tableau (1) we obtain the following allowed tableaux

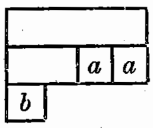
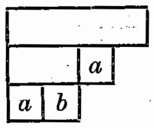
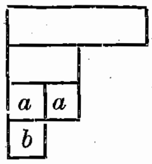


( is not allowed by iv))

from (2), (3), (4), (5) we obtain respectively:

(2) → 

(3) → 

(4) →  (5) →  (6) → 

Representations corresponding to the so-obtained tableaux are then the irreducible components of the tensor product considered (as previously observed we have not to consider tableaux with more than three rows):

$$(2,2)\otimes(1,1) = 27 \otimes 8 = (3,3)+(4,1)+(4,4)+(2,2)+(2,2)+(3,0)+\\ +(0,3)+(1,1).$$

The dimension of $(2,2)\otimes(1,1)$ is equal to $27\cdot 8 = 216$, which using (25) can be checked with the dimension of the second member.
Reader can easily obtain the following decompositions:

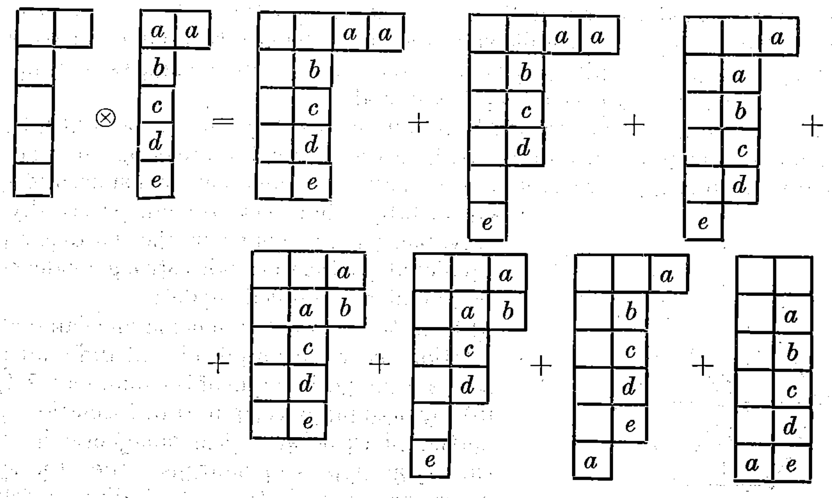
$$8\otimes 8 = 1+8+10+\overline{10}+27$$

$$8\otimes 10 = 35+27+10+8$$

$$8\otimes 10 = 8+\overline{10}+27+35.$$

The same procedure can be applied to the reduction of the tensor product of arbitrary SU_n representations, labeling with $c, d, \ldots$ boxes in third, fourth, .. rows. At the end of course we will drop tableaux containing more than n rows.
Consider the case of SU_6 and the product:

$$35 \otimes 35 = (1, 0, 0, 0, 1) \otimes (1, 0, 0, 0, 1)$$



i.e. $$35 \otimes 35 = (2, 0, 0, 0, 2) + (2, 0, 0, 1, 0) + (1, 0, 0, 0, 1) + (0, 1, 0, 0, 2) + (0, 1, 0, 1, 0) + (1, 0, 0, 0, 1) + (0, 0, 0, 0, 0).$$

10. Eightfold Way

10.1. We are now in position to build up a concrete theory for strongly interacting particles. The first thing to do, after what we have said in sect. 7.5, is to identify particles with the same spin and the same parity with linearly independent vectors inside certain irreducible representations of

$$G \times U_1(N),$$

where G is a Lie group of rank two and $U_1(N)$ is the baryonic number gauge group. This identification fixes the connection between infinitesimal generators of the group and isospin and hypercharge operators.
Of course we must preserve the isospin structure of the particles, in that particles belonging to the same isomultiplet must go into the same irreducible representation.
The procedure is not straightforward since there are three non isomorphic rank two groups, and in addition particles do not exhibit any impressive regularity, apart from the isospin multiplet structure.
Hence at a pure classificatory level one has no clear indication on which the underlying symmetry group, actually is. Furthermore, once a group has been chosen, it is not clear which are the correct assignments of the particles to its irreducible representations.
In fact many different models have been proposed and in principle the right choice should emerge from a comparison of theoretical predictions with experiments.

However the great difficulty is that even large discrepancies could be due not to a substantial failure of the model, but to the effects of the symmetry breaking interactions.

As stressed by A. SALAM [23], it is hoped that a deeper understanding of the latter will finally lead to a non-ambiguous interpretation of many possible tests.

The main informations a symmetry model provides us, are of the following type:

i) whatever the model is, one finds that the known particles alone do not arrange themselves in complete supermultiplets, and one is then led to assume the existence of new particles with definite values of T_3 and Y. One can even guess their masses and the most likely production and decay modes.

However there is no a-priori reason to require completely filled supermultiplets. in fact one can imagine situations in which the symmetry breaking interactions, which are certainly present (see sect. 7.5), make some members of a supermultiplet so unstable, that they are not practically observable. On the contrary the discovery of a predicted particle provides strong evidence in favour of the implied model;

ii) from the symmetry scheme one can deduce relations between amplitudes of different processes; (see the example of iso-spin, sect. 7.4).

iii) by assuming certain transformation properties of weak and electromagnetic interaction Lagrangians of hadrons under the symmetry group, with the aid of Wigner-Eckart theorem (sect. 6.3), one can derive relations between the amplitudes of weak and electromagnetic processes involving such particles.

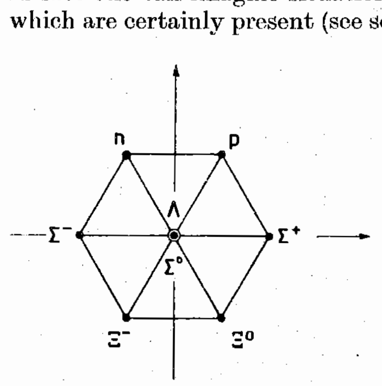


Fig. 6.

In what follows, we will focus our attention on the "eightfold way" model proposed by NE'EMAN [24] and GELL-MANN [3], which has proved to be the most successful one[20].

10.2. a) In the eightfold way model one chooses SU_3 as the underlying symmetry group, and associates the eight "stable" baryons $N, \Sigma, \Lambda^\circ, \Xi$ to the basis vectors of its (1, 1) eight dimensional representation, as indicated in fig. 6.

We can easily deduce the relations between T_3, Y, and the diagonal elements H_1, H_2, by remembering (sect. 9.3b) that in the representation (m, n) of SU_3 the eigenvalues of H_1 and H_2 corresponding to the maximal weight Λ are

$$\Lambda \equiv \left(\frac{m+n}{2\sqrt{3}}, \frac{m-n}{6}\right).$$

Hence, with $m = n = 1$ we have

$$\Lambda \equiv \left(\frac{1}{\sqrt{3}}, 0\right).$$

If we want to associate to $|\Lambda\rangle$ the Σ^+ particle, we have to put

$$T_3 = \sqrt{3}\ H_1 \tag{1}$$

[20]) For a concise discussion concerning the other symmetry models see [25, 26].

and in addition, by requiring p to correspond to $|\Lambda - \alpha_3|\rangle$,

$$Y = 2H_2.$$

The operator T^2 defined in sect 9.3b, which removes all degeneracies inside a supermultiplet, is just the square of the isospin. We will choose normalized simultaneous eigenvectors of T^2, T_3, Y as a basis in any irreducible representation. The pseudoscalar mesons also fit very nicely into an octet according to the scheme reported in fig. 7.

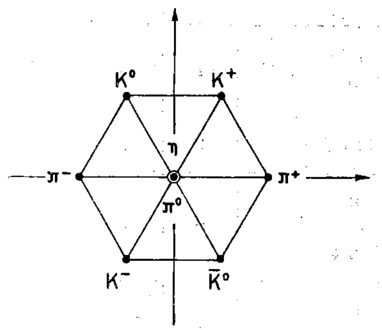


Fig. 7.

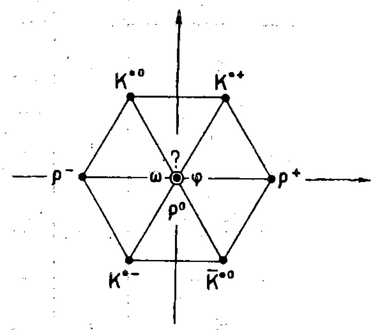


Fig. 8.

It should be noted that the η meson, which quite naturally completes the octet, has been discovered after the introduction of the eighfold way.
When we try to arrange the vector mesons into the scheme, we get in trouble. In fact we would like to assign the nine mesons $\rho(T = 1, Y = 0)$, $K^*(T = 1/2, Y = 1)$, $\bar{K}^*(T = 1/2, Y = -1)$, $\omega = (T = 0, Y = 0)$, $\varphi(T = 0, Y = 0)$ to the same irreducible representation. However, since T_3, T^2, Y are a complete set of commuting operators inside each irreducible representation, it is not possible to fit in the same supermultiplet two distinct isosinglets (ω, φ) with the same hypercharge. The usual assignment is to put eight mesons into an octet (fig. 8) and the remaining one into a singlet (i. e., in the (0, 0) representation), and the question arises whether the ω or the φ particle is to be put in the singlet.

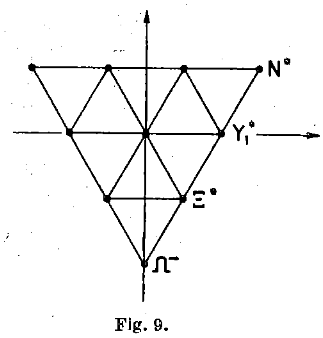


Fig. 9.

We shall see in the following that this question can be consistently resolved.
In considering the baryon-meson resonances, we have to find a representation containing an isospin $^3/_2$ multiplet with hypercharge equal to one, corresponding to the well known $N^*_{3/2}$, $\pi - N$ resonance. The lowest representations containing this isomultiplet are 10 and 27. The first one accomodates wery well the $N = 1$, $J^P = {}^3/_2{}^+$ resonances $N^*_{3/2}$, Y^*_1, Ξ^*, Ω^- (fig. 9).
Whereas $N^*_{3/2}$, Y^*_1, Ξ^* were at hand when it was proposed to assign them to the decuplet, the Ω^- was not yet known: the model predicted its quantum numbers $(N = 1, T = 0, Y = -2)$ as well as its mass $M_\Omega = 1676$ MeV.

On this indications extensive searches for the Ω^- have been carried out, until a particle with right mass and hypercharge has been found studying K^-p reactions at 5.0 GeV/c [27].
The positive result of the search has been considered as one of the most brilliant successes of the eightfold way model.
There is another particle, namely the Y_0^* (mass = 1405 MeV, $T = 0$, $Y = 0$, $J^P = 1/2^-$) which usually is assigned to a (0,0) representation.

Name of the particle	N	J^P	Mass (MeV)	Y	T
N	1	$1/2^+$	(p) 938.2 (n) 939.6	1	$1/2$
Λ^0	1	$1/2^+$	1115.4	0	0
Σ	1	$1/2^+$	+ 1189.4 − 1197.1 1192.4	0	1
Ξ	1	$1/2^+$	− 1321 1314	−1	$1/2$
Y_0^*	1	$1/2^-$?	1405	0	0
N^*	1	$3/2^+$	1236 ± 2	1	$3/2$
Y_1^*	1	$3/2^+$	1382.1 ± 0.9	0	1
Ξ^*	1	$3/2^+$	1529.1 ± 1.0	−1	$1/2$
Ω^-	1	$3/2^+$?	1675 ± 3	−2	0
π	0	0^-	± 139.6 135.0	0	1
η	0	0^-	548.7 ± 0.5	0	0
K	0	0^-	+ 493.8 498.0	1	$1/2$
ρ	0	1^-	763 ± 4	0	1
ω	0	1^-	782.8 ± 0.5	0	0
φ	0	1^-	1019.5 ± 0.3	0	0
K^*	0	1^-	891 ± 1	1	$1/2$

We report a table deduced from [28]: in it we report the quantum numbers (including masses) of the particles we have arranged in the scheme. In the same reference many other particles are listed which we have not considered here mainly for two reasons: first of all the quantum numbers of many of them are at this time not well established (sometimes even their existence is doubtful). Secondly it appears that the situation is so incomplete to make not very useful an even tentative grouping of such particles into supermultiplets.

10.2b) It is convenient for further applications, to identify the eigenstates of T^2, T_3, Y, (to which particles are assigned) inside the tensor representations we have described in sect. 9.6, 9.7.

The particular symmetrical tensors $T^{i_1 \ldots i_m}_{j_1 \ldots j_m}$ defined as

$$T^{i_1 \ldots i_m}_{j_1 \ldots j_n} = \begin{cases} 1 \text{ when } i_1 \ldots i_m \text{ is a permutation of} \\ \underbrace{11 \ldots 1}_{k_1} \ \underbrace{22 \ldots 2}_{k_2} \ \underbrace{33 \ldots 3}_{k_3} \text{ and } j_1 \ldots j_n \text{ is} \\ \text{a permutation of } \underbrace{11 \ldots 1}_{h_1} \ \underbrace{22 \ldots 2}_{h_2} \ \underbrace{33 \ldots 3}_{h_3} \\ 0 \text{ otherwise} \end{cases}$$

are, according to sect. 9.6, eigenstates of H_1, H_2 (inside the representation $(3)^m \otimes (\bar{3})^n$) with weights

$$(k_1 - h_1)\Lambda^1 + (k_2 - h_2)\Lambda^2 + (k_3 - h_3)\Lambda^3 ,$$

where $\Lambda^{1,2,3}$ are the weights of representation 3. Hence they are eigenstates of T_3, Y with eigenvalues

$$T_3 = \frac{1}{2}(k_1 - h_1) - \frac{1}{2}(k_2 - h_2)$$

$$Y = \frac{1}{3}(k_1 - h_1) + \frac{1}{3}(k_2 - h_2) - \frac{2}{3}(k_3 - h_3).$$

They are a basis in the manifold of the tensors which are symmetrical in upper and lower indices, so that to obtain a basis in the (m, n) representation, which diagonalizes T_3 and Y, it is necessary to take those linear combinations of (1) which are traceless. Among them we will select those linear combinations, which correspond to eigenstates of T^2.

In what follows we will be concerned only with the eight and ten dimensional representations.

In the case of the (8) representation we have to consider tensors with two indices. We have not to impose any particular symmetry property, but only the trace condition:

$$T^i_i = 0 .$$

There are nine independent tensors T^i_j:

	T_3	Y
$(T_{(1)})^i_j = \delta^{i1}\delta_{j1}$	0	0
$(T_{(2)})^i_j = \delta^{i1}\delta_{j2}$	1	0
$(T_{(3)})^i_j = \delta^{i1}\delta_{j3}$	$^1/_2$	1

	T_3	Y
$(T_{(4)})^i_j = \delta^{i2}\delta_{j1}$	-1	0
$(T_{(5)})^i_j = \delta^{i2}\delta_{j2}$	0	0
$(T_{(6)})^i_j = \delta^{i2}\delta_{j3}$	$-1/2$	1
$(T_{(7)})^i_j = \delta^{i3}\delta_{j2}$	$-1/2$	-1
$(T_{(8)})^i_j = \delta^{i3}\delta_{j2}$	$1/2$	-1
$(T_{(9)})^i_j = \delta^{i3}\delta_{j3}$	0	$0.$

$T_{(1)}$, $T_{(5)}$, $T_{(9)}$ are not traceless. There are only two independet traceless li nea combinations

$$\begin{aligned} &T_{(1)} - T_{(5)} \\ &T_{(1)} + T_{(5)} - 2T_{(9)} \end{aligned} \tag{2}$$

which span, together with $T_{(2)}$, $T_{(3)}$, $T_{(4)}$, $T_{(6)}$, $T_{(7)}$, $T_{(8)}$ the (1.1) representation. It is worth noticing that the general tensor of (1,1) can be thought as a three by three traceless matrix, and the transformation law 9.6 (20) can be written as a matrix product

$$T \to T' = U T U^{-1} \quad U \in SU_3.$$

Then the operators representing the SU_3 Lie algebra act as

$$(\Lambda T) = [\lambda, T].$$

where λ is any linear combination of the matrices given in sect 9.6 (19). It is then very easy to see that the tensors (2) are eigenstates of T^2 with eigenvalues, respectively, 2 and 0.

From this point of view, recalling what we said in the previous subsection, we see that:

i) in the case of stable baryons, we have the following assignments:

$$\Sigma^+ \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \Lambda^0 \to \begin{Bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{6}} \end{Bmatrix} \quad \text{etc.}$$

Which are symbolically summarized in the matrix:

$$B = \begin{Bmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda^0}{\sqrt{6}} \end{Bmatrix} \tag{3}$$

Factors $1/\sqrt{2}$, $1/\sqrt{6}$ have been introduced so that our basis tensors are normalized with respect to the scalar product

$$(T, T') = \sum_{ij} T^i_j \, (T'^i_j) = \text{Trace}\,[T\,(T')^+]\,.$$

i) In the same way for pseudoscalar mesons we have the matrix:

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \tag{4}$$

For what concerns the 10-dimensional (3,0) representation, we notice that it is spanned by the symmetrical tensors:

$$(T_{\alpha\beta\gamma})^{ijk} = \begin{cases} 1 \text{ when } ijk \text{ is a permutation of } (\alpha, \beta, \gamma = 1,2,3) \\ 0 \text{ otherwise} \end{cases}$$

$\alpha\beta\gamma$	T_3	Y	T^2
111	$^3/_2$	1	$^{15}/_4$
112	$^1/_2$	1	$^{15}/_4$
122	$-^1/_2$	1	$^{15}/_4$
222	$-^3/_2$	1	$^{15}/_4$
113	1	0	2
123	0	0	2
223	-1	0	2
133	$^1/_2$	-1	$^3/_4$
233	$-^1/_2$	-1	$^3/_4$
333	0	-2	0

10.3. Antiparticle multiplets

So far we have not explicity considered the assignments of antiparticles.
If $|a\rangle$ is a particle state, then the corresponding antiparticle state $|\bar{a}\rangle$ has the same space-time properties, but the values of all charges (Q, N, Y, etc.) are reversed. $|a\rangle$ and $|\bar{a}\rangle$ are connected by the charge conjugation operator C

$$|\bar{a}\rangle = C\,|a\rangle,$$

which can be assumed to be a unitary operator satisfying:

$$C^2 = 1\,.$$

It is a well known fact that strong interactions are symmetrical under C, so that if particles exhibit, the SU_3 symmetry, relative to strong interactions the same behaviour must be displayed by the corresponding antiparticles.

In particular for each supermultiplet of baryonic number N, the corresponding antiparticle states must arrange according, to a supermultiplet with baryonic number $-N$. By definition C satisfies

$$\begin{aligned} CQ + QC = \{C,Q\} &= 0 \\ \{C,N\} &= 0 \\ \{C,Y\} &= 0, \end{aligned} \tag{5}$$

so that, using the Gell-Mann-Nishijima formula ($Q = T_3 + {}^1/_2 Y$), we get

$$\{C,T_3\} = 0. \tag{6}$$

Let us indicate with

$$L = \{|(m, nN), T^2, t_3, y\rangle\}$$

the linear manifold spanned by the basis vectors of the (m, n) supermultiplet of particles with baryonic number N particles and with

$$L_C = \{C\,|(m, n\,N)\,T^2, t_3, y\rangle\}$$

the corresponding antiparticle states. Then by (5), (6), we have

$$\begin{aligned} T_3\, C\,|(m, n\,N)\,T^2, t_3, y\rangle &= -t_3\, C\,|(m, n\,N)\,T^2, t_3, y\rangle \\ Y\ C\,|(m, n\,N)\,T^2, t_3, y\rangle &= -y\ C\,|(m, n\,N)\,T^2\,t_3, y\rangle \\ N\ C\,|(m, n\,N)\,T^2, t_3, y\rangle &= -NC\,|(m, n\,N)\,T^2, t_3, y\rangle, \end{aligned} \tag{7}$$

i.e. all the weights of $\{C\,|(m, n\,N)\,T^2, t_3, y\rangle\}$ are opposite to those of $\{|(m, n\,N)$ $T^2, t, y\rangle\}$. From what we said in sect. 9.5 we conclude that antiparticles transform under $SU_3 \otimes U_1(N)$ as members of the (n, m) representation (with baryonic number-N).

In the tensor formalism we have employed before, it is possible to define the C-operation simply as the interchange of lower and upper indices: for example $T^{ij} \xrightarrow{c} T_{ij}$.

We note that in general this is a mapping between two different tensor spaces, unless the number of upper and lower indices are equal, i.e. the representation is self-conjugate.

In the case of antibaryons we have the following assignments:

$$B \equiv \begin{pmatrix} \dfrac{\bar{\Sigma}^0}{\sqrt{2}} + \dfrac{\bar{\Lambda}^0}{\sqrt{6}} & \bar{\Sigma}^- & \bar{\Xi}^- \\ \bar{\Sigma}^+ & -\dfrac{\bar{\Sigma}^0}{\sqrt{2}} + \dfrac{\bar{\Lambda}^0}{\sqrt{6}} & \bar{\Xi}^0 \\ \bar{p} & \bar{n} & -\dfrac{2\bar{\Lambda}^0}{\sqrt{6}} \end{pmatrix} \tag{8}$$

(the bar indicates that these particles have baryonic number -1).

The pseudoscalar meson multiplet is mapped into itself by C since mesons have baryonic number equal to zero:

$$M \to (\bar{M})^T = \begin{pmatrix} \frac{\bar{\pi}^0}{\sqrt{2}} + \frac{\bar{\eta}}{\sqrt{6}} & \bar{\pi}^- & \dots \\ \bar{\pi}^+ & \dots\dots & \\ \dots\dots & & \end{pmatrix}$$

In the case of vector mesons a minus sign appears due to the fact that neutral vector mesons have negative charge conjugation.

10.4. Mass formulae

Till now we have grouped particles into supermultiplets, without worrying about the very large mass difference involved. The mass of a particle is the mean value of its Hamiltonian H in its rest frame: if SU_3 were a symmetry group in the strict sense, H would be an invariant operator so that, by Schur's lemma, inside each irreducible representation, H would be a multiple of the unit operator, i.e. particles inside the same multiplet would have the same mass. This is not the case and, as stressed before, we are forced to assume that a component of H violates unitary symmetry. We write

$$H = H_0 + H_1,$$

where H_0 is invariant under SU_3. For what concerns H_1, it must be charge independent and hypercharge conserving

$$[H_1, T_i] = [H_1, Y] = 0, \quad i = 1, 2, 3$$

so that H_1 is constant inside each isomultiplet (we do not consider at this point electromagnetic and weak contributions to the mass).

Let us label each strongly interacting particle in its rest frame with the quantum numbers $N, J^P, \lambda, T_3, T^2, Y$, where λ is the label of the SU_3 representation to which it is assigned. Physical masses are then the eigenvalues of the matrix:

$$\langle N', J'^{P'}, \lambda', T^2, T'^2, Y' |H| N, J^P, \lambda, T_3, T^2, Y\rangle.$$

We have written N and J^P for completeness but obviously H commutes with them, so that the relevant matrix elements are:

$$\langle NJ^P, \lambda', T_3', T'^2, Y' |H| N, J^P, \lambda, T_3, T^2, Y\rangle = \langle \lambda' T_3' T'^2 Y' |H| \lambda T_3 T^2 Y\rangle =$$
$$= \langle \lambda' T_3' T'^2 Y' |H_0| \lambda T_3 T^2 Y\rangle + \langle \lambda' T_3' T'^2 Y' |H_1| \lambda T_3 T^2 Y\rangle. \quad (9)$$

To simplify notations we assume that particles with the particular values of N and J^P considered, group together in only two supermultiplets λ and λ'. This case easily extends to the general one, and in addition this is the most complicated situation that has been found till now.

Using the assumptions made on H_0 and H_1, the matrix element (9) writes as:

$$m_0(\lambda)\, \delta_{\lambda\lambda'}\, \delta_{T^2T'^2}\, \delta_{YY'}\, \delta_{T_3T_3'} + m_1(T^2, Y)_{\lambda\lambda'}\, \delta_{T^2T'^2}\, \delta_{YY'}\, \delta_{T_3T'_3}. \quad (10)$$

Hence H_0 is diagonal as it has to be. Matrix elements of H_1 do not depend on T_3 so that we will not menti on it further on. Moreover H_1 can connect only states belonging to isomultiplets with the same T^2 and Y. Recalling what we said in sect. 9.3b, we see that in λ as well as in λ' isomultiplets with given values of T^2 and Y can occur at most once, so that the matrix representing H_1, has the form:

$$\begin{array}{cc} & \begin{array}{cc} T^2, Y & T^2, Y \end{array} \\ \begin{array}{c} \lambda \left\{ T^2, Y \right. \\ \\ \lambda' \left\{ T^2, Y \right. \end{array} & \left(\begin{array}{ccccc} \cdots & \vdots & \cdots & \vdots & \cdots \\ \cdots & m_1(T^2 Y)_{\lambda\lambda} & \cdots & m_1(T^2 Y)_{\lambda\lambda'} & \cdots \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ \cdots & m_1(T^2 Y)_{\lambda'\lambda} & \cdots & m_1(T^2 Y)_{\lambda'\lambda'} & \cdots \\ \cdots & \vdots & \cdots & \vdots & \cdots \end{array} \right) \\ & \begin{array}{cc} \underbrace{\qquad\qquad\qquad}_{\lambda} & \underbrace{\qquad\qquad\qquad}_{\lambda'} \end{array} \end{array} \tag{11}$$

To diagonalize this matrix is equivalent to diagonalize each submatrix

$$\begin{pmatrix} m_1(T^2, Y)_{\lambda\lambda} & m_1(T^2, Y)_{\lambda\lambda'} \\ m_1(T^2, Y)_{\lambda'\lambda} & m_1(T^2, Y)_{\lambda'\lambda'} \end{pmatrix} \tag{12}$$

for each value of T^2 and Y occurring both in λ and λ'.

In the case of stable baryons, pseudoscalar mesons, and decuplet resonances, for each value of N and J^P, only one representation is present, so that H_1 has the form

$$H_1 \equiv \begin{pmatrix} m_1(T^2, Y) & & 0 \\ & m_1(T^2, Y') & \\ 0 & & \ddots \end{pmatrix} \tag{13}$$

and each isomultiplet has the mass

$$m = m_0 + m_1(T^2, Y).$$

We have found nothing else but that particles with same T^2, Y have the same mass. Significant results are obtained by assuming H_1 to be the $Y = 0$, $T_3 = 0$, $T^2 = 0$ member of a set of tensor irreducible operators transforming as the regular SU_3 representation.

In a field theoretical treatment one would describe the symmetry breaking interactions by adding to the symmetrical Lagrangian $\mathcal{L}_0$ a term $\mathcal{L}_{\text{MS}}$ which is required to be hypercharge and isospin conserving:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{MS}}.$$

Using perturbation theory in $\mathcal{L}_{\text{MS}}$ one would write mass corrections as the expectation value of a power serie in $\mathcal{L}_{\text{MS}}$. If we assume $\mathcal{L}_{\text{MS}}$ to transform like the $T^2 = T_3 = Y = 0$ member of an octet, we are led in first order approximation to our assumption on H_1.

With this proviso we can obtain the general form of matrix elements (13), using Wigner-Eckart theorem (sect. 6.3.).
We write H_1 as $T^{(8)}_{0,0,0}$ and introduce the notation

$$C(\lambda, T^2, T_3, Y;\ 8, 0, 0, 0;\ \mu_\gamma T^2 T_3 Y)$$

for the Clebsch-Gordan coefficient connecting the vectors

$$|\lambda, T^2 T_3 Y\rangle\ |8, 0, 0, 0\rangle;\quad |\mu_\gamma T^2 T_3 Y\rangle,$$

where the latter vectors span the standard basis which decomposes the tensor product $\lambda \otimes 8$. The suffix γ distinguishes between equivalent representations appearing in the decomposition.
Then, according to Wigner-Eckart theorem, we have:

$$m = m_0 + m_1(T^2, y) = m_0 + \sum_\gamma C(\lambda, T^2, T_3, Y; 8, 0, 0, 0; \mu_\gamma T^2 T_3 Y) \cdot \langle \lambda \| T^{(8)} \| \mu_\gamma \rangle. \tag{14}$$

Consider stable baryons. In this case $\lambda = 8$, and $8 \otimes 8 = 1 + 8 + 8 + 10 + \overline{10} + 27$, so that the regular representation appears twice in this reduction, and we have two reduced matrix elements.
Using the Clebsch-Gordan coefficients reported in [*29*] one finds:

$$\begin{aligned} m_N &= m_0 - \frac{\sqrt{5}}{10} \langle 8 \| T^{(8)} \| 8_1 \rangle + \frac{1}{2} \langle 8 \| T^{(8)} \| 8_2 \rangle \\ m_\Xi &= m_0 - \frac{\sqrt{5}}{10} \langle 8 \| T^{(8)} \| 8_1 \rangle - \frac{1}{2} \langle 8 \| T^{(8)} \| 8_2 \rangle \\ m_\Lambda &= m_0 - \frac{\sqrt{5}}{5} \langle 8 \| T^{(8)} \| 8_1 \rangle \\ m_\Sigma &= m_0 + \frac{\sqrt{5}}{5} \langle 8 \| T^{(8)} \| 8_1 \rangle . \end{aligned} \tag{15}$$

We have four masses and three unknown parameters. Their elimination leads to the relation (firstly given by GELL-MANN):

$$m_N + m_\Xi = \frac{3}{2} m_\Lambda + \frac{1}{2} m_\Sigma \tag{16}$$

(2256 MeV) (2268 MeV).

The values of the masses have been taken from the table listed before.
We apply the same procedure to the decuplet resonances. Since in the reduction $8 \otimes 10 = 8 + 10 + 27 + 35$ the 10 representation appears only once, in the formula (14) we have only one reduced matrix element.
Using again Clebsch-Gordan coefficients, one obtains:

$$M_{N^*} = M_0 - \frac{1}{\sqrt{8}} \langle 10 \| T^8 \| 10 \rangle;\ M_{Y_1^*} = M_0;\ M_{\Xi^*} = M^0 + \frac{1}{\sqrt{8}} \langle 10 \| T^{(8)} \| 10 \rangle;$$

$$M_{\Omega^-} = M_0 + \frac{2}{\sqrt{8}} \langle 10 \| T^{(8)} \| 10 \rangle .$$

Since there are two unknown parameters and four masses, one can obtain:

$$
\begin{array}{c}
M_{N^*} - M_{Y^*} = M_{\Xi^*} - M_{\Omega^-} \\
(146 \pm 3\ \text{MeV}) \quad (146 \pm 4\ \text{MeV}) \\
2 M_{\Xi^*}\, M = {}_{Y_1^*} + M_{\Omega^-} \\
(2058 \pm 2\ \text{MeV}) \quad (2057 \pm 3\ \text{MeV}).
\end{array} \tag{17}
$$

These formulae simply state that masses in the decuplet are equally spaced.
The mass of Ω^- can be obtained from the known masses of N^* and Y_1^*. This predicted value, as we said, is amazingly close to the experimental one.
For the octet of the pseudoscalar mesons the argument is the same as that made for the stable baryons. Thus we get a formula analogous to (16):

$$
\begin{array}{c}
m_K + m_{\overline{K}} = \frac{3}{2} m_\eta + \frac{1}{2} m_\pi \\
(992\ \text{MeV}) = (891\ \text{MeV}).
\end{array} \tag{18}
$$

Two remarks are in order:
1) $m_K = m_{\overline{K}}$ by charge-conjugation invariance of strong interactions.
2) (18) is in very poor agreement with experiment.
Substantial improvement can be achieved if we put in (18) the squared masses instead of the masses:

$$
\begin{array}{c}
2 m_K^2 = \frac{3}{2} m_\eta^2 + \frac{1}{2} m_\pi^2 \\
(49.2 \cdot 10^4\ (\text{MeV})^2) \quad (46.2 \cdot 10^4\ (\text{MeV})^2).
\end{array} \tag{19}
$$

An argument given by Feynman to support this substitution is that in any field theoretical model, corrections to bare boson masses affect directly m^2.
The same remarks apply to the vector mesons which will be treated in next section.

10.5. $\omega - \varphi$ mixing

We have previously assigned eight vector mesons to an octet and the ninth to an SU_3 singlet; but we had no way to decide whether the ω or the φ had to be placed in the singlet. With the aid of the mass formula for the octet, which has of course the same form as (19), we find:

$$
2 m_{K^*}^2 = \frac{3}{2} m_8^2 + \frac{1}{2} m_\rho^2 \tag{20}
$$

(m_8 = mass of the isosinglet i.e. of the $T^2 = Y = 0$ member of the octet)
Inserting the known values for m_{K^*}, m_ρ, one obtains $m_8 = 930 \pm 3$ MeV. This value is intermediate between $m_\varphi = 1019, 5 \pm 3$ MeV and $m_\omega = 782. \pm 0.5$ MeV; however both differences $m_\varphi^2 - m_8^2 = 18 \cdot 10^4\ \text{MeV}^2$, $m_8^2 - m_\omega^2 = 25 \cdot \cdot 10^4\ \text{MeV}^2$ are very large.
If we insist that the mass formula must hold even in this case to an accuracy comparable to that obtained for the three other cases considered, we must con-

clude that neither ω nor φ can be identified sic et simpliciter with the isosinglet of the octet.
However physical particles are eigenstates of:

$$H = H_0 + H_1.$$

And H in the representation $8 \oplus 1$ to which the vector mesons are assigned is represented by the matrix:

$$H = \begin{pmatrix} M_0^2 + (K^* | H_1 | K^*) & 0 & 0 & 0 & 0 \\ 0 & M_0^2 + (\rho | H_1 | \rho) & 0 & 0 & 0 \\ 0 & 0 & M_0^2 + (\overline{K^*} | H_1 | \overline{K^*}) & 0 & 0 \\ 0 & 0 & 0 & M_0^2 + (8 | H_1 | 8) & (8 | H_1 | 1) \\ 0 & 0 & 0 & (1 | H_1 | 8) & M_1^2 + (1 | H_1 | 1) \end{pmatrix}$$

If now M_0 is nearly equal to M_1, we can expect that the off-diagonal elements of H are important, so that the $T = 0$, $Y = 0$ eigenstates of H which we want to identify with ω and φ, are considerably different from the corresponding eigenstates of H_0.
We have then to diagonalize the submatrix

$$\begin{pmatrix} M_0^2 + \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & M_1^2 + \varepsilon_{22} \end{pmatrix} \tag{21}$$

where

$$\varepsilon_{11} = \langle 8 | H_1 | 8 \rangle, \; \varepsilon_{12} = \langle 8 | H_1 | 1 \rangle \text{ etc.}$$

Changing, if necessary, phases of the states $|8\rangle$, $|1\rangle$ we can take ε_{12} real so that from the hermiticity of H_1, $\varepsilon_{12} = \varepsilon_{21}$.
We write:

$$\begin{aligned} |\varphi\rangle &= \cos\theta \, |8\rangle + \sin\theta \, |1\rangle. \\ |\omega\rangle &= -\sin\theta \, |8\rangle + \cos\theta \, |1\rangle. \end{aligned} \tag{22}$$

The angle θ must be determined requiring $|\varphi\rangle$ and $|\omega\rangle$ to be eigenstates of (21) with eigenvalues equal to their physical masses squared [*30*].
From this one finds:

$$(\tan\theta)^2 = \frac{m_\varphi^2 - m_8^2}{m_8^2 - m_\omega^2} = 0{,}69; \; \theta = 39°50' \tag{23}$$

where $m_8^2 = M_0^2 + \varepsilon_{11}$ is the value given by the mass formula for the octect.
According to these results φ and ω appear to be superpositions of pure SU_3 states, in contrast to all the other particles we have considered [*31*].
It is important to note that, in contrast to the other cases, the introduction of the mixing angle θ rises to four the number of parameters needed to describe the vector mesons mass spectrum $(\theta, M_0, M_1, \langle 8 | H_1 | 8 \rangle)$. Having four masses at our disposal $m_{K^*}, m_\rho, m_\omega, m_\varphi$ it is not possible in this context to have any test of the theory. The possibility of determining θ in an independent way will be discussed later.

Our derivation of the sum rules (16), (17), (19), 20), was based on the knowledge of the Clebsch-Gordan coefficients involved.
Alternatively one can find the general structure inside each irreducible representation of an operator, such as H_1, commuting with T_1, T_2, T_3, Y and transforming according to the regular representation of SU_3.
Such an operator must have the form:

$$H_1 = f(T^2, Y) = a\,1 + b\,Y + c\,T^2 + d\,Y^2 + \cdots.$$

The stated transformation property further restricts H_1 to the form

$$H_1 = b\,Y + c\left(T^2 - \frac{1}{4}\,Y^2\right) + a' \cdot 1,$$

so that in each irreducible representation the Hamiltonian is

$$H = H_0 + H_1 = a \cdot 1 + b\,Y + c\left(T^2 - \frac{1}{4}\,Y^2\right). \tag{24}$$

For fermions (24) gives the mass formula:

$$M = a \cdot 1 + b\,Y + c\left(T\,(T + 1) - \frac{1}{4}\,Y^2\right). \tag{25}$$

For bosons, taking into account conditions $CHC^{-1} = H$, $C\,Y\,C^{-1} = -\,Y$ (where C is the charge-conjugation operator), b must vanish:

$$m^2 = a + c\left(T(T + 1) - \frac{1}{4}\,Y^2\right). \tag{26}$$

For the decuplet resonance one has the relation $T = 1/2\,Y + 1$, which reduces (26) to

$$m = a' + b\,Y, \tag{27}$$

giving the equal spacing rule.
The general formula (24) has been given by S. OKUBO [32]

10.6. Baryon-meson Yukawa couplings

It is interesting to find out all the Yukawa-type couplings between baryons and pseudoscalar mesons which are invariant under SU_3, In fact we will find that these couplings involve only two constants so that writing them in the usual isotopic spin form, one can derive relations between coupling constants such as $g_{NN\pi}$, $g_{\Sigma\Lambda\pi}$, $g_{N\Xi K}$ etc.
In a field-theoretical model one assumes baryons and mesons fields to behave under SU_3 as tensor operators (see sect. 6.3) belonging to the eight dimensional representation.
The Yukawa-type interaction Lagrangian has the form

$$\mathcal{L} = \sum_{\alpha\beta\gamma} g_{\alpha\beta\gamma}\,\bar{B}_\alpha \gamma_5\,B_\beta\,M_\gamma$$

where B_β and $\bar{B}_\alpha$ are, respectively, baryon and antibaryon fields specified by the labels β and α (i.e. p, n, ... $\bar{\text{p}}$, $\bar{\text{n}}$, ...) and M_γ is a meson field specified by γ (i.e. $\pi^+, \pi^-, \ldots$).
Invariance under SU_3 demands $\mathcal{L}$ to transform as a scalar operator. Now the operators

$$\bar{B}_\alpha \gamma_5 B_\beta M_\gamma$$

transform like members of the $8 \otimes 8 \otimes 8$ representation, and obviously the same holds for $\mathcal{L}$. Coefficients $g_{\alpha\beta\gamma}$ have to be determined by requiring $\mathcal{L}$ to be one of those vectors which in the decomposition of $8 \otimes 8 \otimes 8$ belong to the irreducible (0, 0) components. Using the method described in sect. 9.7 one can see that there are two (0, 0) components, i.e. only two possible invariant Yukawa-couplings.
According to sect. 10.3, antibaryon fields transform as

$$(\bar{B}) \to U \bar{B} U^{-1} \qquad U \in SU_3$$

i.e. just like baryons, due to the fact that the 8-representation is selfconjugate. It is then easy to verify that the tensor operator

$$\text{Trace}\,(\bar{B}\gamma_5\, B M) \tag{28}$$

(trace involves summation only over SU_3 indices, so that γ_5 must be treated as a number) is invariant. In the same way one can see that also the operator

$$\text{Trace}\,(\bar{B}\gamma_5 M B) \tag{29}$$

is invariant.
Instead of (28) and (29) we will use the so called F and D combinations defined as

$$(F)\ \text{Trace}\,(\bar{B}\gamma_5[B, M]); \quad (D)\ \text{Trace}\,(\bar{B}\gamma_5\{B, M\}),$$

which are obviously linearly independent invariant operators. Recalling that in the product $8 \otimes 8 \otimes 8$ there are exactly two operators of such kind, we conclude that the most general invariant trilinear operator in mesons, baryons and antibaryons fields is a linear combination of F and D. In particular:

$$\mathcal{L} = \sum_{\alpha\beta\gamma} g_{\alpha\beta\gamma}\, \bar{B}_{\alpha\gamma 5}\, B_\beta M_\gamma = g_F\, \text{Tr}\,(\bar{B}\gamma_5\, [B, M]) + g_D\, \text{Tr}\,(\bar{B}\gamma_5\, \{B, M\}). \tag{30}$$

To deduce couplings in terms of isomultiplets one has merely to substitute (3), (4), (8) into (30), to carry out the trace and to rearrange terms in order to have a combination of isospin couplings (for example of the form $g_{\text{NN}\pi}\, \overline{\boldsymbol{N}}\boldsymbol{\tau}\, \boldsymbol{N}\boldsymbol{\pi}$, $\overline{\boldsymbol{\Sigma}}\, \boldsymbol{\Sigma}\boldsymbol{\eta}$ etc.).
If we define:

$$g_{\text{NN}\pi} = g = \frac{g_D - g_F}{\sqrt{2}}$$

$$\alpha = \frac{-g_F}{\sqrt{2}\, g},$$

then all coupling constants can be expressed in terms of α and g. We refer the reader to [*33*] for a complete list of these relations.

The experimental situation does not allow a precise determination of α. The value $\alpha = 1$ (i.e. pure F-type coupling) seems to be excluded by hyperfragments binding, which requires $g_{\Lambda\Sigma\pi} = 2/\sqrt{3}\, g\,(1-\alpha) \neq 0$. The dynamical calculations of MARTIN and WALI [34] indicate $0.15 \leq \alpha \leq 0.56$ i.e. a prevalence of the D-type coupling. With this set of values $g_{N\Lambda K}$ turns out to be of the same order of magnitude than $g_{NN\pi}$. This seems to be in contrast with K-photoproduction data, which suggest $g_{N\Lambda K}$ an order of magnitude smaller than $g_{NN\pi}$.
These discrepancies can be in principle accounted for by symmetry breaking interactions. One other possibility has been pointed out in [35].
Same considerations can be applied to the baryon-vector meson couplings, leading to two possible Lagrangians: one F-type, and the other D-type.
If one writes the F-type coupling in terms of isospin multiplets, [36] the ρ appears to be coupled to the isospin current (i.e. to terms like $\bar{N}\boldsymbol{\tau}N, \bar{\boldsymbol{\Sigma}}\times\boldsymbol{\Sigma}$, etc.) and the ω_8 (i.e. the $T_3 = Y = T = 0$ member of the vector meson-octet) to the hypercharge current (i.e. to the term $\bar{N}N + \bar{\Xi}\Xi$) whereas in the D-type these peculiar couplings do not appear. Now we know (for example from the isovector part of the electromagnetic form factors of the nucleon) that ρ is actually coupled to the isospin current, so that in this case we have to assume only F-type coupling. The interaction Lagrangian then is

$$\mathcal{L} = g_{BBV}\,\mathrm{Tr}\,(B\gamma_\mu\,[B,\, V_\mu]),$$

where V_μ is a matrix analogous to (4), with the substitutions:

$$\pi \to \rho_\mu$$
$$K \to K^*_\mu$$
$$\eta \to (\omega_8)_\mu$$

We have now only one parameter.
The vector meson singlet (i.e. the φ_0) is coupled with the baryonic number current:

$$\mathcal{L} = g_{BBV}\,(\varphi_0)_\mu\,\mathrm{Tr}\,(\bar{B}\gamma_\mu B)$$

Consider now the pseudoscalar-vector meson couplings: also in this case we may have two invariant combinations of terms like

$$\left(\frac{\partial}{\partial x_\mu} M_\alpha\right) M_\beta\,(V_\mu)_\gamma$$

(α, β, γ, are SU_3 indices which label the various mesons.), namely

$$\mathrm{Trace}\ (V_\mu M\,(\partial_\mu M) - V_\mu\,(\partial_\mu M)\,M) = Tr\,(V_\mu[M, \partial_\mu M])$$
$$\mathrm{Trace}\ (V_\mu M\,\partial_\mu M + V_\mu \partial_\mu M M) = Tr\,(V_\mu\,\{M, \partial_\mu M\}).$$

However the second term, by virtue of the condition:

$$\partial_\mu V_\mu = 0,$$

is equivalent to a divergence, i.e. it adds a divergence to the Lagrangian, so that it can be assumed to vanish. In fact:

$$V_\mu M(\partial_\mu M) + V_\mu(\partial_\mu M)M = V_\mu \partial_\mu(MM) = \partial_\mu(V_\mu MM) + (\partial_\mu V_\mu)MM =$$
$$= \partial_\mu(V_\mu MM).$$

Hence also this coupling is pure F. In contrast with the previous case the vector meson singlet cannot be coupled to pseudoscalar mesons: in fact the only possible coupling would be

$$(\varphi_0)_\mu \operatorname{Tr}((\partial_\mu M)M),$$

but under charge conjugation $M \to M^T$, $\partial_\mu M \to \partial_\mu M^T$, $\varphi_{0\mu} \to -\varphi_{0\mu}$ (charge conjugation of ρ_0, ω, and φ is -1) so that this coupling is not invariant under C. This has the consequence that the decay

$$\varphi_0 \to \mathrm{K} + \overline{\mathrm{K}}$$

is forbidden, whereas

$$\omega_8 \to \mathrm{K} + \overline{\mathrm{K}}$$

is allowed, i.e. only the componend of the φ particle on the octet can decay in in $\mathrm{K}\overline{\mathrm{K}}$. This fact can be used in principle to determine the $\omega - \varphi$ mixing independently from mass formulae [37].

10.7 Decuplet decays

The decays of the baryon decuplet resonances allowed by energy, T_3, T^2 and Y conservation are

$$\mathrm{N}^* \to \mathrm{N} + \pi; \quad \mathrm{Y}_1^* \to \begin{cases} \Lambda^0 \pi \\ \Sigma\ \pi \end{cases}; \quad \Xi^* \to \Xi\pi$$

(Ω^- is stable against electromagnetic and strong decays because of its mass, which is less than the threshold of the $\Xi\overline{\mathrm{K}}$ channel which is the only open for these interactions). These decays, in the limit of exact SU_3, are described by a single amplitude. In fact in this case the matrix elements involved are of the type

$$M(\mathrm{N}^* \to \mathrm{N}\pi) = (\mathrm{N}^* \,|S|\, \mathrm{N}\pi), \tag{31}$$

where S is scalar under SU_3. By reducing the product $8 \otimes 8$, with Clebsch-Gordan coefficients, we can write (31) as the matrix element of a scalar operator between vectors of irreducible representations,
Such matrix elements are zero for vectors belonging to irreducible components of $8 \otimes 8$ different from 10, whereas they are equal to

$$a\,\delta_{T^2 T'^2}\,\delta_{T_3 T_3'}\,\delta_{YY'}$$

for vectors of the 10 component. Hence we have

$$M(N^{*++} \to p + \pi^+) = -\frac{1}{\sqrt{2}}a; \quad M(Y_1^{*+} \to \Lambda\pi) = -\frac{1}{2}a$$

$$M(Y_1^{*+} \to \Sigma_0\pi^+) = \frac{1}{\sqrt{12}}a; \quad M(Y_1^{*+} \to \Sigma^+\pi^0) = -\frac{1}{\sqrt{12}}a \tag{32}$$

$$M(\Xi^{*0} \to \Xi^- + \pi^+) = -\frac{1}{\sqrt{6}}a; \quad M(\Xi^{*0} \to \Xi^0\pi^0) = \frac{1}{2\sqrt{3}}a,$$

(all the other amplitudes can be obtained from these by isotopic spin symmetry) where factors multiplying a are the proper Clebsch-Gordan coefficients.
If we want to compare (32) with experiments we have to take into account mass differences. The most simple thing to do is to introduce mass differences into the phase space factors which multiply $|M|^2$ in the expression for rates, leaving untouched relations (32).
Predictions so obtained are in an unpleasant disagreement with experiments. For example one has:

$$\frac{\text{Rate } (Y_1^{*+} \to \Sigma\pi)}{\text{Rate } (Y_1^{*+} \to \Lambda\pi)} = (\text{phase space ratio}) \times \frac{4}{6} = \frac{P_\Sigma^3}{P_\Lambda^3} \cdot \frac{2}{3} \simeq 12\%$$

$(p_{\Sigma,\Lambda} =$ momentum of Σ or Λ particle),

whereas experimentelly the ratio is consistent with zero ($\sim 2 \pm 2\%$).
This discrepancy can be thought as due to large non symmetrical interactions, which must be properly accounted for.
In fact it has been shown by V. GUPTA and V. SINGH [*38*] and by C. BECCHI, E. EBERLE, G. MORPURGO that inserting a symmetry breaking interaction of the type used for mass formulae, one can derive relations between decuplet decay amplitudes which agree well with the experimental data.
10.8. The main test of the unitary symmetry model in strong interactions would be to check experimentally the relations which one can derive between amplitudes of different scattering processes. However relations obtained assuming full SU_3 symmetry widely disagree with experimental data [*23*] and again one has to take into account the role of symmetry breaking interactions. This role has not been till now satisfactorily understood so that, partly for this reason, partly for lack of experimental data, we do not know at present how to make meaningful tests of the eightfold way with scattering processes.

11. Electromagnetic Interactions

11.1. We know that the electromagnetic field interacts with hadrons in such a way to conserve T_3 and Y; moreover, due to the smallness of the coupling constant, we can describe such interaction with a pertubative method, starting from an interaction Lagrangian of the form:

$$\mathcal{L}_{\text{int}} = e j_\mu(x) A^\mu(x). \tag{1}$$

Here $j_\mu(x)$ is the electromagnetic current of hadrons. The structure of this operator depends upon the dynamics of strong interactions themselves, which we at present do not know in detail.
Hence the Lagrangian (1) has to be considered as a phenomenological device which allows us to explicitly consider the dependence on electromagnetic field of these interactions, whereas unknown effects of strong interactions are lumped into the local operator $j_\mu(x)$. We note that the matrix elements of $j_\mu(x)$ between physical states are connected to measurable quantities (form factors).
By definition we have

$$\int j_4(x)\, d^3x = Q \tag{2}$$

and by charge, Y, T_3 conservation:

$$\partial^\mu j_\mu(x) = 0$$

$$[j_\mu(x), T_3] = [j_\mu(x), Y] = 0. \tag{3}$$

In the context of the eightfold way model, relations between matrix elements of $j_\mu(x)$ can be obtained by the knowledge of the commutation relations between $j_\mu(x)$ and the generators of SU_3.
From the Gell-Mann Nishijima relation we have:

$$\int j_4(x)\, d^3x = Q = T_3 + \frac{1}{2} Y = 3H_1 + H_2.$$

This suggests $j_\mu(x)$ to be composed of two parts: the current of the third component of total isospin plus one half the hypercharge current

$$j_\mu(x) = j_\mu^{(T_3)}(x) + \frac{1}{2}\, j_\mu^{(Y)}(k), \tag{4}$$

and now $j_\mu^{(T_3)}(x)$ and $j_\mu^{(Y)}(x)$ have the same transformation properties (under SU_3) as T_3 and Y. From this it is easy to see that $j_\mu(x)$ commutes with $E_{\pm 3}$ and the same for $\mathcal{L}_{\text{int}}$. If we put (see sect. 8.7.c)

$$U_3 = \frac{3}{4} Y - \frac{1}{2} T_3 = \frac{3}{2} H_2 - \frac{\sqrt{3}}{2} H_1$$

$$U_\pm = \sqrt{3}\, E_{\mp 3},$$

we see that U_3, $U_\pm$ have the same commutation relations of the isotopic spin generators, and are called the U-spin generators. Hence electromagnetic interactions at all perturbative orders conserve charge and U-spin, which in this case play the same role as hypercharge and I-spin for medium strong interactions. From the previous analysis we see that everything we said for medium strong interactions can be applied to electromagnetic interactions, substituting the partition into I-spin and Y-multiplets of irreducible SU_3 representations with a partition into U-spin and Q-multiplets.
We give for reference the decomposition of the pseudoscalar mesons octet into U-spin multiplets (Fig. 10).

One sees that π^+, K^+ and K^-, π^- constitute two U-spin doublets whereas K_0 and $\bar{K}_0$ are members of an U-spin triplet. π^0 and η are eigenstates of U_3, with eigenvalue zero, but they are not eigenstates of U^2. Instead the combinations

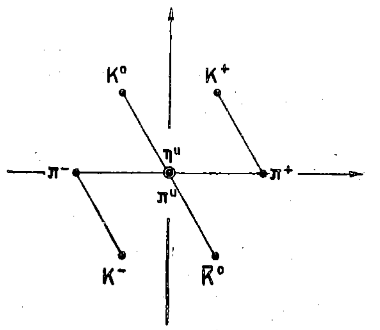


Fig. 10. U-spin multiplets in the pseudoskalar mesons octet

$$|\pi^u\rangle = \frac{1}{2}\,|\pi^0\rangle - \frac{\sqrt{3}}{2}\,|\eta\rangle$$

$$|\eta^u\rangle = \frac{\sqrt{3}}{2}\,|\pi^0\rangle + \frac{1}{2}\,|\eta\rangle$$

are eigenstates of U^2 with eigenvalues 2 and 0, so that π^u, K^0, $\bar{K}^0$ constitute an U-spin triplet and η^u is an U-spin singlet.
As a general rule in the weight diagram U-spin multiplets are directed along the α_3 root.
From what we said, we can draw very easily a certain number of consequences:
a) *Electromagnetic mass splittings* — In calculating mass splittings due to symmetry breaking interactions we neglected the electromagnetic effects, which we want now to take into account.
We start from a situation in which, apart from SU_3-invariant strong interactions, only electromagnetic interactions are present. In this case the masses of the particles inside an SU_3 multiplet split up according to U-spin multiplets, being (1) invariant under U-spin. Hence masses obey a law of the form

$$m = m_0 + m(Q, U) \qquad (m_0 = \text{common mass of the } SU_3 \text{ multiplet}).$$

In the case of stable baryons we get in particular the relations:

$$m_p = m_{\Sigma^+}$$
$$m_n = m_{\Xi^0} \tag{5}$$
$$m_{\Xi^-} = m_{\Sigma^-}.$$

Relations (5) are of course not satisfied by the actual masses; this is natural because we have neclected the important contribution of medium strong interactions. We can however deduce from (5) the relation:

$$m_n - m_p = (m_{\Xi^0} - m_{\Xi^-}) - (m_{\Sigma^+} + m_{\Sigma^-}) \tag{6}$$

which has been given firstly by S. COLEMAN and S. L. GLASHOW [*41*].
If we now turn on the medium strong symmetry breaking interactions, masses of particles lying inside the same I-spin multiplet are shifted of the same amount, so that we way expect relation (6) to be left unchanged, since it compares mass differences for particles with same T^2. In fact inserting the experimental values (6) reads:

$$(m_{\Sigma^-} - m_{\Sigma^+}) - (m_n - m_p) = 6.38 \pm 0.3\ \text{MeV}$$

$$m_{\Xi^-} - m_{\Xi^0} = 6.5 \pm 1.2\ \text{MeV (data from [28])}.$$

The agreement is excellent. One should remark that the previous derivation of (6) rests on the possibility of treating in an independent way the medium strong and electromagnetic effects. This seems not to be in general a legitimate procedure. In a field theoretical treatment we would write a Lagrangian made up of three terms:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{MS} + \mathcal{L}_{e.m.}$$

$\mathcal{L}_0$ is the symmetrical part, whereas $\mathcal{L}_{MS}$ and $\mathcal{L}_{e.m.}$ are responsible of medium strong and electromagnetic interactions. Using perturbative methods, the correction to the symmetrical mass m_0 is expressed as the expectation value of a power series in $\mathcal{L}_{MS} + \mathcal{L}_{e.m.}$. Hence we see that COLEMAN and GLASHOW relation follows if we neglect all terms which contain powers of the product $\mathcal{L}_{MS}\mathcal{L}_{e.m.}$, retaining all orders in $\mathcal{L}_{MS}$ and $\mathcal{L}_{e.m.}$.
From this point of view the validity of (6) is quite unexpected (see however [*23*]).
b) *Magnetic moments* — Just in the same way, neglecting medium strong interactions and assuming only U-spin and charge conservations, we obtain the following relations between magnetic moments of stable baryons [*39, 40*] (the same relations apply to the electric and magnetic form factors)

$$\begin{aligned} \mu(\Sigma^+) &= \mu(\mathrm{p}) \\ \mu(\Sigma^-) &= \mu(\Xi^-) \\ \mu(\Xi^0) &= \mu(\mathrm{n}) \\ -\frac{1}{\sqrt{3}}\,\mu(\Sigma^0\Lambda) &= \mu(\mathrm{n}) - \mu(\Lambda) \\ -\sqrt{3}\,\mu(\Lambda\Sigma^0) &= \mu(\mathrm{n}) - \mu(\Sigma^0), \end{aligned} \tag{7}$$

where $\mu(\Sigma^0\Lambda)$ is called the transition magnetic moment between Σ^0 and Λ, and appears for example in the amplitude of the decay $\Sigma^0 \to \Lambda + \gamma$.
The experimental information available up to now does not allow to test any-one of the relations (7).
For pseudoscalar mesons we obtain easily the results [*40*] that the form factors of K^+ and π^+ are equal, whereas the form factors of K^0 and $\bar{K}^0$ are zero.
In fact:

form factor (K^0) = form factor ($\bar{K}^0$) by U-spin
form factor (K^0) = — form factor ($\bar{K}^0$) by Charge-conjugation

c) η *and* π^0 — *two photons decay* — We said previously that the $U_3 = Q = 0$ eigenstates of U^2 are (in the pseudoscalar meson octet)

$$|\pi^u\rangle = \frac{1}{2}\,|\pi^0\rangle - \frac{\sqrt{3}}{2}\,|\eta\rangle \quad (\mathrm{U} = 1)$$

$$|\eta^u\rangle = \frac{\sqrt{3}}{2}\,|\pi^0\rangle + \frac{1}{2}\,|\eta\rangle \quad (U = 0),$$

conversely one has:

$$|\pi^0\rangle = \frac{1}{2}\,|\pi^u\rangle + \frac{\sqrt{3}}{2}\,|\eta^u\rangle$$

$$|\eta\rangle = -\frac{\sqrt{3}}{2}\,|\pi^u\rangle + \frac{1}{2}\,|\eta^u\rangle.$$

Observe now that $|\pi^u\rangle$ cannot decay, by U-conservation, into two photons (which have $U = 0$) so that the amplitudes for $(\pi^0, \eta) \to 2\gamma$ are equal to

$$A(\pi^0 \to 2\gamma) = \frac{\sqrt{3}}{2} A(\eta^u \to 2\gamma)$$

$$A(\eta \to 2\gamma) = \frac{1}{2} A(\eta^u \to 2\gamma),$$

i.e.

$$A(\pi^0 \to 2\gamma) = \sqrt{3}\, A(\eta \to 2\gamma).$$

After phase space corrections, assuming the lifetime of π^0 to be equal to $1.5 \cdot 10^{-16}$s. we obtain a width for $\eta \to 2\gamma$ of 140 eV [*41*], which is not inconsistent with present data.

d) *First order relations* — Up to this point we have only used U-spin invariance. More detailed informations can be obtained if we retain in the perturbative expansions considered only first order terms (in the electromagnetic coupling) i.e. terms containing the elctromagnetic current only once.

In this case we can exploit the assumption contained in formula (4), i.e. the fact that $j_\mu(x)$ transforms as a member of the eight-dimensional representation of SU_3. By using Wigner-Eckart theorem, for example, we can express all magnetic moments of stable baryons in terms of only two magnetic moments; (the same applies to electric form factors). For example we find

$$\mu_\Lambda = \frac{1}{2}\mu_n = -0.95 \text{ nuclear magnetons},$$

whereas experimentally

$$\mu_\Lambda = -0.66 \pm 0.35 \text{ nuclear magnetons}.$$

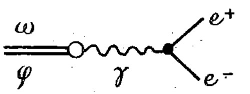
All the other explicit relations are contained in [*40*].

Finally, let us consider the electromagnetic decays:

$$\omega \to e^+ + e^-(\mu^+ + \mu^-)$$

$$\varphi \to e^+ + e^-(\mu^+ + \mu^-).$$

Both decays can be thought to go through the one-photon channel [*37*]



(8)

In the amplitude for the process (8) the matrix element of the electromagnetic current between the ω (or φ) state and the vacuum is involved. Now if we write (sect. 10.5 (22))

$$|\varphi\rangle = \cos\theta\,|8\rangle + \sin\theta\,|1\rangle$$

$$|\omega\rangle = \sin\theta\,|8\rangle + \cos\theta\,|1\rangle,$$

we see that we have to evaluate the matrix elements:

$$\langle 1|\, j_\mu(x)\,|0\rangle$$

$$\langle 8|\, j_\mu(x)\,|0\rangle.$$

$j_\mu(x)$ transforms as the 8 representation and, whereas the product $8 \otimes 8$ contains the singlet representation (to which the vacuum is assigned) this representation is not contained in the product $8 \otimes 1$. Hence, by Wigner-Eckart theorem, the first matrix element vanishes. We conclude that an SU_3 singlet cannot decay through the one photon channel, so that only the components of ω and φ over the octet can go (at first order) into $e^+ + e^-$ $(\mu^+ + \mu^-)$.
The ratio

$$\frac{\Gamma(\varphi \to e^+e^-)}{\Gamma(\omega \to e^+e^-)} = \text{(phase space corrections)} \tan^2\theta$$

provides in principle a measure of $\tan^2\theta$ independent from mass formulae.
Up to now however only the $\omega \to e^+ + e^-$ decay has been observed, so that we cannot make any comparison of the theory with experiment.

12. Leptonic Decays of Hadrons

12.1. Very exciting results have been obtained by the application of SU_3 symmetry to the field of weak interactions of hadrons. We will not give here an extensive discussion of all the topics involved, limiting ourselves to sketch the theory for leptonic decays[21]). These processes have the general form

$$A \to l + \nu_l + B + B' + \cdots$$
$$A \to l + \nu_l,$$

where $A, B, B' \ldots$ are strongly interacting particles, l is a lepton (e, μ), ν_l the corresponding neutrino. A few significant examples are:

		ΔT	ΔS	ΔQ	
$\Delta S = 0$	$\pi^+ \to \pi^0 + e^+ + \nu_e$	-1	0	-1	
	$\pi^+ \to \mu^+ + \nu_\mu$	-1	0	-1	
	$n \to p + e^- + \bar{\nu}_e$	1	0	1	
	$\Sigma^+ \to \Lambda + e^+ + \nu_e$	-1	0	-1	(1)
$\Delta S = 0$	$K^+ \to \pi^0 + e^+ + \nu_e$	$-1/2$	-1	-1	
	$K^+ \to \mu^+ + \nu_\mu$	$-1/2$	-1	-1	
	$\Lambda \to p + e^- + \bar{\nu}_e$	$1/2$	1	1	
	$\Sigma^- \to n + e^- + \bar{\nu}_e$	$1/2$	1	1	

All these processes can be described starting from an interaction Lagrangian of the form:

$$\mathcal{L}_I = \frac{G}{\sqrt{2}} [J_\mu (j_\mu)^+ + \text{H. c.}] \tag{2}$$

J_μ and j_μ are the weak currents associated to hadrons and leptons, and G is the weak coupling constant determined from μ-decay.

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[21]) For a more detailed treatment of weak interactions see [*42, 43, 44*].

For what concerns j_μ, expermental findings agree with the form (in terms of lepton fields)

$$j_\mu = \bar{e}\gamma_\mu(1+\gamma_5)\nu_e + \bar{\mu}\gamma_\mu(1+\gamma_5)\nu_\mu, \tag{3}$$

whereas the structure of J_μ, which is thought to be determined by strong interactions, is not known in detail. The amplitudes of the processes we are considering are expressed as matrix elements of (2)

$$\frac{G}{\sqrt{2}}\left[\langle B + B' + \ldots | J_\mu | A\rangle \langle l\,\nu_l | j_\mu^+ | 0\rangle + \langle BB' \ldots | J_\mu^+ | A\rangle \langle l\,\nu_l | j_\mu | 0\rangle\right];$$

the matrix elements of j_μ can be calculated, so that the actual difficultyis constituted by the other terms.

For what concerns space-time properties, experiments indicate that J_μ can be splitted up in two terms: one transforming as a vector and the other as a pseudovector: we will refer to them as to the vector and axial-vector currents:

$$J_\mu = J_\mu^V + J_\mu^A .$$

The first one is responsible e.g. for the β-decay of π^+:

$$\pi^+ \to \pi^0 + e^+ + \nu_e ;$$

and the other for the usual π decay:

$$\pi^+ \to \mu^+ + \nu_\mu .$$

Let us define ΔS, ΔT_3, ΔQ respectively as the changes of strangeness, isospin, charge suffered by hadrons[22]) (see (1)). Then we can divide leptonic decays into two classes: $\Delta S = 0$, and $\Delta S \neq 0$ decays. Experiments indicate [43] that the following selection rules are satisfied within errors (which are however rather large):

i) for $\Delta S = 0$ decays, $|\Delta T_3| = 1$ (hence $\Delta Q = \pm 1$)

ii) for $\Delta S \neq 0$ decays $\Delta S = \Delta Q$, $|\Delta S| = 1$ (hence $|\Delta T_3| = 1/2$).

This selection rule forbids for example the process:

$$\Sigma^+ \to n + e^+ + \nu_e$$

which actually has not been seen [45].

We are then led to write J_μ as:

$$J_\mu = J_\mu^{V(0)} + J_\mu^{V(1)} + J_\mu^{A(0)} + J_\mu^{A(1)}, \tag{4}$$

where $J_\mu^{V(0)}$ is the strangeness conserving and $J_\mu^{V(1)}$ is the $\Delta S = 1$ part of the vector current, and the same for axial currents.

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[22]) These changes are not independent. From the Gell-Mann Nishijima formula:

$$\Delta Q = \Delta T_3 + \frac{1}{2}\Delta S.$$

In the context of isotopic spin symmetry one could attempt to explain selection rule i), assuming the $\Delta S = 0$ current to tranform under SU_2 as a tensor operator, belonging to the isospin one representation.
The CVC (conserved vector current) theory of FEYNMANN and GELL-MANN [46] embodies this assumption in a much stronger statement; they identify the $\Delta S = 0$ vector currents $J_\mu^{V(0)}$, $(J_\mu^{V(0)})^+$ with the $T_3 = \pm 1$ components of the isospin current, i.e. of the current which arises from SU_2 invariance (see sect. 7.3.).
The $T_3 = 0$ component is the isovector part of the electromagnetic current. As a consequence, insofar we neglect electromagnetic effects, $J_\mu^{V(0)}$ and $(J_\mu^{V(0)})^+$ are conserved:

$$\partial_\mu J_\mu^{V(0)} = 0, \; \partial_\mu (J_\mu^{V(0)})^+ = 0 .$$

We quote three significant examples of predictions made by CVC theory [42]:
a) the agreement between the Fermi constant in β-decay of nuclei and the Fermi constant in μ-decay;
b) the rate of the pion β-decay can be calculated from the neutron β-decay obtaing a result in agreement with experiments:
c) using Wigner-Eckart theorem one can express the matrix elements of $J_\mu^{V(0)}$, $(J_\mu^{V(0)})^+$, for example between nucleon states, in terms of the electromagnetic current matrix elements, i.e. of nucleon e. m. form factors which are known from $e - N$ scattering experiments. Preliminary data on neutrino experiments seem to support this prediction.
The $\Delta S = 0$ axial current is also assumed to transform as an isovector, but in contrast to $J_\mu^{V(0)}$ it is not conserved.
The simplest generalization to include strange particles decays is to assume J_μ^V and J_μ^A to possess well-defined transformation properties under SU_3. In particular if we assume J_μ^V and J_μ^A to belong to octets of tensor operators, it is obvious that selection rules i), ii) are fulfilled (of course the converse is not true: i) ii) do not imply octet currents).
Now SU_3 symmetry provides us an octet of vector currents (see sect. 7.3) which, in absence of symmetry breaking interactions, are conserved. We could then identify various pieces of J_μ^V with such currents. This however would imply each current to be coupled to leptons with the same strenght, i.e. with the same coupling constant as the $\Delta S = 0$ part. On the contrary experiments give coupling constants for strange particle decays which are smaller of an order of magnitude than the $\Delta S = 0$ couplings.
N. CABIBBO has assumed [47—48] that the vector current coupled to leptons has the form

$$J_\mu^V = \cos\theta \, J_\mu^{V(0)} + \sin\theta \, J_\mu^{V(1)}, \tag{5}$$

where $J_\mu^{V(0)}$, $(J_\mu^{V(0)})^+$, $J_\mu^{V(1)}$, $(J_\mu^{V(1)})^+$, are the $T_3 = \pm 1$, $Y = 0$, $T_3 = \pm 1/2$, $Y = \pm 1$ members of the octet of currents deriving from SU_3 invariance (to which electromagnetic current belongs, sect. 11.1). θ is an angle which characterizes weak interactions of all hadrons. Moreover the axial current is assumed to have the form

$$J_\mu^A = \cos\theta \, J_\mu^{A(0)} + \sin\theta \, J_\mu^{A(1)} \tag{6}$$

with the same θ as (5). $J_\mu^{A(0)}$, $(J_\mu^{A(0)})^+$, $J_\mu^{A(1)}$, $(J_\mu^{A(1)})^+$ are tensor operators transforming as the $T_3 = \pm 1$, $Y = 0$, $T_3 = \pm 1/2$, $Y = \pm 1$ members of an octet.

We give here a brief account of predictions which can be obtained from this theory in the case of baryons decays. For a detailed discussion see [*43, 49*].
Matrix elements of J_μ^V between baryon states can be expressed in terms of θ and of two reduced matrices ($8 \otimes 8$ contains the eight representation twice). By analogy with sect. 10.6 let us call them F_V, D_V. In the limit of zero momentum transfer, they can be written as:

$$(F_V)_\mu = \bar{u}(p_f)\gamma_\mu F_V(k^2)\,u(p_i)$$
$$(D_V)_\mu = \bar{u}(p_f)\gamma_\mu D_V(k^2)\,u(p_i) \tag{7}$$

p_i, p_f: baryons initial and final momentum
$k = p_f - p_i$, $k^2 \to 0$.
And in the limit in which such currents are conserved

$$F_V(0) = 1, \quad D_V(0) = 0,$$

so that the matrix elements of J_μ^V are determined by θ.
For what concerns axial currents, again we have two reduced matrices, F, D which in the same limit as (7) can be written as

$$F_\mu = \bar{u}(p_f)F(k^2)\gamma_\mu\gamma_5 u(p_i) \; (k^2 \to 0)$$
$$D_\mu = \bar{u}(p_f)\,D(k^2)\gamma_\mu\gamma_5\,u(p_i), \tag{8}$$

but now the lack of conservation of J_μ^A does not allow to obtain additional conditions on $F(0)$, $D(0)$.
Concluding we see that the baryons decays (in the limit $k^2 \to 0$, absence of symmetry breaking interactions[23]) are described in terms of three numbers: $F(0)$, $D(0)$, θ.
H. COURANT et al. [*45*] find two sets of parameters consistent with data, both with near the same value of θ, but differing for the ratio F/D.
It is remarkable that near the same value of θ ($\theta \simeq 0.25$) has been given by CABIBBO in the paper previously quoted, comparing the decays:

$$K^+ \to \mu^+ + \nu_\mu$$
$$\pi^+ \to \mu^+ + \nu_\mu .$$

In these decays only axial currents contribute ($(J_\mu^{A(1)})^+$ and $(J_\mu^{A(0)})^+$) respectively) so that the branching ratio:

$$\frac{R(K^+ \to \mu^+ + \nu_\mu)}{R(\pi^+ \to \mu^+ + \nu_\mu)}$$

is proportional to

$$\left| \frac{\langle K^+ | (J^{A(1)})^+ | 0 \rangle}{\langle \pi^+ | (J^{A(0)})^+ | 0 \rangle} \right|^2 ,$$

i.e. to $\tan^2 \theta$.

---

[23]) When the symmetry breaking interaction is taken into account at the first order, it can be shown that our conclusions on the vector current remain correct [*50*].

13. Concluding Remarks

13.1. The idea of a higher symmetry, in the concrete formulation of the eightfold way model, undoubtely greatly improves the phenomenological description of the behaviour of strongly interacting particles. However the very fact that it works rises the need of understanding at a deeper, dynamical level, how the symmetry is brought about (as well as its partial violation).
For an up-to-date discussion of the various attempts made in this direction, using bootstrap technique as well as field theoretical methods, see [23]. To the latter class belongs SCHWINGER's W_3 model [51] as well as the popular "quarks" or "aces" model (proposed by ZWEIG and GELL-MANN].
Another interesting problem is that of the connection between internal and space-time symmetries.
In this context very promising is the SU_6 model proposed by GÜRSEY, RADICATI and PAIS [52]; see also [53]) who treat on the same footing spin, isospin and hypercharge.

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